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Fermion representation of the rolling tachyon boundary conformal field theory

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Abstract: A free fermion representation of the rolling tachyon boundary conformal field theory is constructed. The representation is used to obtain an explicit, compact, exact expression for the boundary state. We use the boundary state to compute the disc and cylinder amplitudes for the half-S-brane.

Keywords: Conformal Field Models in String Theory, Bosonic Strings, D-branes, Tachyon Condensation.

Contents

1. Introduction **1**

2. Preliminaries **3**
- 2.1 Bosonization and fermionization 3
- 2.2 Boundary states 8
- 2.3 Open string partition functions 9

3. Rolling tachyon and fermionization **11**
- 3.1 Fermion representation of the doubled bosons 13
- 3.2 Quantum theory of the bosons (ϕ_1, ϕ_2) 14
- 3.3 Current algebra 17

4. Exact boundary state for the unstable D-brane **18**
- 4.1 Some simple boundary states 18
- 4.2 Open string partition functions 21
- 4.3 Exact boundary state for the rolling tachyon 23

5. Open string partition function for the rolling tachyon **23**

6. Disc amplitude and time evolution of the rolling tachyon **24**
- 6.1 Momentum states 25
- 6.2 Rolling tachyon 27

7. Conclusion **28**

A. Cocycles for fermion operators **29**
- A.1 One dimensional system 29
- A.2 Two dimensional system 31

1. Introduction

One of the most important puzzles in string theory is the fate of unstable D-branes and how to describe their time evolution toward that fate. An intriguing conjecture for their time evolution is due to Ashoke Sen [1] and is called the rolling tachyon [2]. It asserts the existence of an exact time-dependent classical solution of string theory describing the decay of a D-brane by the condensation of open string tachyons.

In the sigma model approach to the boson string, the rolling tachyon is obtained by adding an exactly marginal boundary operator to the world-sheet action,

$$S = -\frac{1}{4\pi}\int_M d\tau d\sigma \partial X^0 \cdot \partial X^0 + \oint_{\partial M} d\sigma \left(Ae^{X^0} + Be^{-X^0} \right) + \cdots . \tag{1.1}$$

We use units where $\alpha' = 1$. The second term is the boundary interaction corresponding to the tachyon field, with A and B constants determined by the initial conditions. The remaining terms denoted by three dots are the action of a $c = 25$ conformal field theory and ghosts.

In the idealized situation, where the tachyon condensate is space-independent, the other degrees of freedom decouple from the time component X^0 and we can study it by itself. We shall assume that this is the case, and will not discuss the other coordinates or ghosts further in this Paper. Note the negative signature of the kinetic term for X^0 arising from the fact that it is the time coordinate.

The simplest choice for the tachyon profile in (1.1) is $A = g/2$ and $B = 0$. With this profile the conformal field theory (1.1) describes an unstable D-brane in the past which then decays into a more stable configuration. In this Paper, we will focus on this configuration, sometimes called the half-S-brane. The profile originally discussed by Sen [1] has $A = B = g/2$ is termed the full-S-brane [3].

To analyze this theory, we begin with a Wick rotation to euclidean time, $X^0 \to iX^0$. After Wick rotation, the interaction in (1.1) is a periodic function of X^0. (From now on we will drop the superscript 0.) It describes a non-compact boson with a periodic boundary potential.

The conformal field theory of a massless scalar field living on a strip with an exactly marginal periodic boundary interaction (with $A = B$) at one boundary and Dirichlet condition at the other boundary was discussed a decade ago in refs. [4, 5]. In ref. [5] it was shown that the boundary state can be given as an exact sum of reparametrization invariant Ishibashi states [6, 7] of the SU(2) current algebra.

Sen observed that this boundary state can be used to describe the rolling tachyon [1]. Since then, a large body of work has explored the implications of this exact, time dependent background [1, 3, 8−39]. It has greatly improved our understanding of the time dependent processes driven by the tachyon instability in string theory. However, our understanding of the final fate of the unstable D-brane and details of the dynamics of the rolling tachyon process is still far from complete. The exact time dependent description is only available for the first few lower levels and a complete description of the time evolution of the boundary state is still lacking.

In this Paper we shall discuss a free fermion representation of the boundary conformal field theory of the rolling tachyon. Our representation is a generalization of one which was proposed by Polchinski and Thorlacius [40]. Like ref. [4, 5], they studied a two-dimensional scalar conformal field theory on a strip with a marginal periodic interaction at one boundary and Dirichlet condition at the other boundary. In order to find a free fermion representation of the conformal field theory, they had to introduce an extra boson. The advantage of the resulting fermionic representation is that the boundary interaction

JHEP05(2005)072

becomes a simple fermion current operator which is bilinear in fermion fields. The result is a free field theory of fermions interacting with a boundary potential. Ref. [40] used it to compute the boundary S-matrix and the partition function which they found to agree with results of the current algebra technique [5].

Here we shall extend the fermion conformal field theory formulated in ref. [40] for the open string to the closed string sector. We will discuss the construction of Neumann, Dirichlet and rolling tachyon boundary states in the fermion language. We will find simple closed form expressions for them. In order to demonstrate the equivalence of the fermion and boson approaches, we use the fermion boundary state to compute the disc amplitude and find agreement with known results. We also use it to compute the cylinder amplitude.

One interesting feature of the closed string sector is that, to obtain all of the discrete momentum and wrapped sectors of the periodically identified boson, we must consider states from the Hilbert spaces of both Ramond and Neveu Schwarz fermions.

2. Preliminaries

Before we go on to discuss the fermion representation of the rolling tachyon in later sections, we pause to review some of the essentials of bosonization and fermionization in this section. We will discuss the representation of a closed string boson variable by fermions. We will show how to find boundary states in terms of fermion variables and we will compute partition functions for closed and open strings. This is a warmup exercise for the slightly more complicated fermionization of the rolling tachyon system which will appear in later sections.

2.1 Bosonization and fermionization

In two dimensional conformal field theory, fermions and bosons are mapped to each other by

$$\psi_L(z) \;=\; e^{-i\frac{\pi}{2}p_R} \;:e^{-i\sqrt{2}\phi_L(z)}:, \qquad \psi_L^\dagger(z) \;=\; e^{i\frac{\pi}{2}p_R} \;:e^{i\sqrt{2}\phi_L(z)}: \tag{2.1a}$$

$$\psi_R(\bar z) \;=\; e^{-i\frac{\pi}{2}p_L} \;:e^{i\sqrt{2}\phi_R(\bar z)}:, \qquad \psi_R^\dagger(\bar z) \;=\; e^{i\frac{\pi}{2}p_L} \;:e^{-i\sqrt{2}\phi_R(\bar z)}: \tag{2.1b}$$

where $\phi_R(\bar z)$ and $\phi_L(z)$ are the right- and left-moving boson fields, respectively. The exponentials $e^{\pm i\frac{\pi}{2}p_{L,R}}$ are cocycles, necessary to make $\psi_R(\bar z)$ and $\psi_L(z)$ anti-commute. (See appendix for more about cocycles.) Here p_L and p_R are moment and the normal ordering is defined below.

The complex number z is the coordinate of the complex plane. The coordinates of the Euclidean cylinder, (τ,σ), are related to it by the conformal map $z = e^{\tau+i\sigma}$, $\bar z = e^{\tau-i\sigma}$. The left- and right-moving boson operators are defined by the mode expansions:

$$\phi_L(\tau+i\sigma) = \frac{1}{\sqrt{2}}x_L - \frac{i}{\sqrt{2}}p_L(\tau+i\sigma) + \frac{i}{\sqrt{2}}\sum_{n\neq 0}\frac{\beta_n}{n}e^{-n(\tau+i\sigma)}\,, \tag{2.2a}$$

$$\phi_R(\tau-i\sigma) = \frac{1}{\sqrt{2}}x_R - \frac{i}{\sqrt{2}}p_R(\tau-i\sigma) + \frac{i}{\sqrt{2}}\sum_{n\neq 0}\frac{\tilde\beta_n}{n}e^{-n(\tau-i\sigma)}\,. \tag{2.2b}$$

JHEP05(2005)072

The non-vanishing commutators are

$$[x_L, p_L] = i\,, \qquad [x_R, p_R] \;=\; i \tag{2.3a}$$

$$[\beta_m, \beta_n] = m\delta_{m+n}\,, \qquad \left[\tilde{\beta}_m, \tilde{\beta}_n\right] \;=\; m\delta_{m+n}. \tag{2.3b}$$

A representation of this algebra begins with a vacuum $|p_L, p_R\rangle$ which is an eigenstate of momenta (p_L, p_R) and is annihilated by positively moded oscillators

$$\beta_k|p_L, p_R\rangle = 0\,, \qquad \tilde{\beta}_k|p_L, p_R\rangle = 0\,, \quad k > 0\,. \tag{2.4}$$

Excited states are created by negatively moded oscillators,

$$(\beta_{-1})^{k_1}(\beta_{-2})^{k_2}\ldots(\tilde{\beta}_{-1})^{\bar{k}_1}(\tilde{\beta}_{-2})^{\bar{k}_2}\ldots|p_L, p_R\rangle\,, \qquad k_n, \bar{k}_n = 0, 1, 2, \ldots$$

Normal ordering of any operator puts all negatively moded operators to the left of positively moded operators.

The equal-time commutation relation in Euclidean space-time is

$$\left[\phi(\tau, \sigma), \partial_\tau\phi(\tau, \sigma')\right] = 2\pi\delta(\sigma - \sigma')\,, \tag{2.5}$$

where

$$\phi(\tau, \sigma) = \phi_L(\tau + i\sigma) + \phi_R(\tau - i\sigma)\,. \tag{2.6}$$

The commutation relations and equation of motion

$$\left(\frac{\partial^2}{\partial\tau^2} + \frac{\partial^2}{\partial\sigma^2}\right)\phi = 0$$

which is solved by the decomposition (2.6) are obtained by canonical quantization of the field theory with action

$$S = \frac{1}{4\pi}\int d\tau d\sigma\; \partial_a\phi\; \partial_a\phi\,. \tag{2.7}$$

Fermions have the mode expansion

$$\psi_L(\tau + i\sigma) = \sum_n \psi_n e^{-n(\tau+i\sigma)}\,, \qquad \psi_L^\dagger(\tau + i\sigma) = \sum_n \psi_n^\dagger e^{-n(\tau+i\sigma)} \tag{2.8a}$$

$$\psi_R(\tau - i\sigma) = \sum_n \tilde{\psi}_n e^{-n(\tau-i\sigma)}\,, \qquad \psi_R^\dagger(\tau - i\sigma) = \sum_n \tilde{\psi}_n^\dagger e^{-n(\tau-i\sigma)} \tag{2.8b}$$

with non-vanishing anticommutation relations

$$\left\{\psi_m, \psi_n^\dagger\right\} = \delta_{m+n}\,, \qquad \left\{\tilde{\psi}_m, \tilde{\psi}_n^\dagger\right\} = \delta_{m+n}\,. \tag{2.9}$$

We must distinguish between the cases when the fermions have periodic or anti-periodic wave-functions on the cylinder. When they are anti-periodic their mode expansion has oscillators ψ_n labeled by n which are half-odd-integers, $(n \in \mathbb{Z} + 1/2)$ so that $(\psi_L, \psi_R) \to (-\psi_L, -\psi_R)$ when $\sigma \to \sigma + 2\pi$. The oscillators can be uniquely separated into those with positive and negative modes. A representation of (2.9) is found by beginning with the

JHEP05(2005)072

vacuum state which is annihilated by all positively moded oscillators

$$\left.\begin{array}{l}\psi_n\,|0\rangle_{NS}=0,\ \ \tilde\psi_n\,|0\rangle_{NS}=0\\ \psi_n^\dagger\,|0\rangle_{NS}=0,\ \ \tilde\psi_n^\dagger\,|0\rangle_{NS}=0\end{array}\right\}\quad n>0 \tag{2.10}$$

and creating excited states by operating on the vacuum with negatively moded oscillators

$$\begin{aligned}&\left(\psi_{-1/2}\right)^{k_{1/2}}\left(\psi_{-3/2}\right)^{k_{3/2}}\cdots\left(\tilde\psi_{-1/2}\right)^{k'_{1/2}}\left(\tilde\psi_{-3/2}\right)^{k'_{3/2}}\cdots\left(\psi^\dagger_{-1/2}\right)^{\bar k_{1/2}}\left(\psi^\dagger_{-3/2}\right)^{\bar k_{3/2}}\cdots\\ &\cdots\left(\tilde\psi^\dagger_{-1/2}\right)^{\bar k'_{1/2}}\left(\tilde\psi^\dagger_{-3/2}\right)^{\bar k'_{3/2}}\cdots|0\rangle\end{aligned} \tag{2.11}$$

with

$$k_n,\bar k_n,k'_n,\bar k'_n=0,1\,.$$

The fermion theory with anti-periodic boundary conditions and half-integrally moded oscillators is called the Neveu-Schwarz (NS) sector. When both left- and right-moving fermions are antiperiodic it is called the NS-NS sector.

When the boundary condition for the fermions is periodic, the oscillators are labeled by integers ($n\in\mathbb{Z}$). Then, the oscillator with $n=0$ is a fermion zero mode. Because of the presence of this zero mode, the states are degenerate. We begin with the state $|--\rangle$ defined by

$$\left.\begin{array}{l}\psi_n\,|--\rangle=0\ ,\ \tilde\psi_n\,|--\rangle=0\quad n\geq 0\\ \psi_n^\dagger\,|--\rangle=0\ ,\ \tilde\psi_n^\dagger\,|--\rangle=0\quad n>0\end{array}\right\}. \tag{2.12}$$

Then, the degenerate vacuum states are $|++\rangle=\psi_0^\dagger\tilde\psi_0^\dagger|--\rangle$, $|+-\rangle=\psi_0^\dagger|--\rangle$, $|-+\rangle=\tilde\psi_0^\dagger|--\rangle$, $|--\rangle$. Excited states are created from the four vacuum states by operating with negatively moded oscillators.

A fermion with periodic boundary conditions is called a Ramond (R) fermion and the theory which we are discussing where both left- and right-moving fermions are periodic is called the R-R sector.

The non-vanishing equal-time anti-commutation relations are

$$\left\{\psi_L(\tau,\sigma),\psi_L^\dagger(\tau,\sigma')\right\}=2\pi\delta(\sigma-\sigma')\,,\qquad\left\{\psi_R(\tau,\sigma),\psi_R^\dagger(\tau,\sigma')\right\}=2\pi\delta(\sigma-\sigma')\,. \tag{2.13}$$

The fermions with this anti-commutation relation and the equations of motion

$$\left(\partial_\tau+i\partial_\sigma\right)\psi_L=0\,,\qquad\left(\partial_\tau-i\partial_\sigma\right)\psi_R=0$$

result from quantization of the field theory with Euclidean action

$$S=\frac{1}{2\pi}\int d\tau d\sigma\ \left[\psi_L^\dagger\left(\partial_\tau+i\partial_\sigma\right)\psi_L+\psi_R^\dagger\left(\partial_\tau-i\partial_\sigma\right)\psi_R\right]. \tag{2.14}$$

We shall see in the following that we shall need to use both R-R and NS-NS fermions.

The fermion theory produces states where the momenta p_L and p_R have certain discrete values. To see this, we recall that momenta are related to fermion numbers. The number operators are obtained by integrating

$$:\psi_L^\dagger(z)\psi_L(z):\ =\ i\sqrt{2}\partial_\tau\phi_L(z)\,,\qquad:\psi_R^\dagger(\bar z)\psi_R(\bar z):\ =\ -i\sqrt{2}\partial_\tau\phi_R(\bar z) \tag{2.15}$$

which can be gotten from eqs. (2.1a), (2.1b) and the operator product expansion. The momenta are then

$$p_L = \int_0^{2\pi} \frac{d\sigma}{2\pi} : \psi_L^\dagger(z)\psi_L(z) : , \qquad p_R = -\int_0^{2\pi} \frac{d\sigma}{2\pi} : \psi_R^\dagger(\bar z)\psi_R(\bar z) : . \tag{2.16}$$

In our quantization of the fermions, the fermion number operators have integer spectra in the NS sector and, because of the fermion zero modes, half-odd-integer spectra in the R sector. In the NS-NS-sector

$$p_L = \sum_{n=\frac{1}{2}}^{\infty} \left(\psi_{-n}^\dagger \psi_n - \psi_{-n}\psi_n^\dagger\right), \qquad p_R = -\sum_{n=\frac{1}{2}}^{\infty} \left(\tilde\psi_{-n}^\dagger \tilde\psi_n - \tilde\psi_{-n}\tilde\psi_n^\dagger\right) \tag{2.17}$$

and in the R-R sector

$$p_L = \sum_{n=1}^{\infty} \left(\psi_{-n}^\dagger \psi_n - \psi_{-n}\psi_n^\dagger\right) + \psi_0^\dagger\psi_0 - \frac{1}{2}, \qquad p_R = -\sum_{n=1}^{\infty} \left(\tilde\psi_{-n}^\dagger \tilde\psi_n - \tilde\psi_{-n}\tilde\psi_n^\dagger\right) - \tilde\psi_0^\dagger\tilde\psi_0 + \frac{1}{2}. \tag{2.18}$$

Consistent with this, in the formula (2.1a), for example, the momentum and coordinate appear as

$$\psi_L = e^{-i\frac{\pi}{2}p_L} e^{-ix_L - p_L\tau - ip_L\sigma + \cdots} . \tag{2.19}$$

Then, unraveling operator ordering gives

$$\psi_L(\tau, \sigma + 2\pi) = -e^{-2\pi i p_L}\psi_L(\tau,\sigma) .$$

From this, we see that, to get anti-periodic NS fermions, the momentum p_L should be quantized as integers, whereas to get periodic R fermions, it should be quantized as half-odd-integers.

The fact that the fermion theory produces only states where the momenta are quantized in either integer or half-odd-integer units means that they can only correspond to some of the states of the boson theory where, for a non-compact boson, the momentum is a continuously varying quantum number. On the other hand, this changes when the boson is periodically identified. Then, when the spatial coordinate of the boson is periodic, the momentum is discrete, and we might hope to match the discrete momenta of compact boson with fermion numbers in the fermion theory. Indeed, the periodicity of the boson should be compatible with the bosonization formulae, (2.1a) and (2.1b).

We can see this more clearly by comparing partition functions of the two theories. The hamiltonian in the boson and fermion representations are

$$L_0^B = \frac{1}{2}p_L^2 + \sum_{n=1}^{\infty}\beta_{-n}\beta_n - \frac{1}{24} \ , \quad \tilde L_0^B = \frac{1}{2}p_R^2 + \sum_{n=1}^{\infty}\tilde\beta_{-n}\tilde\beta_n - \frac{1}{24} , \tag{2.20a}$$

$$L_0^{NS} = \sum_{n\in\mathbb{Z}+1/2} n : \psi_{-n}^\dagger\psi_n : -\frac{1}{24} \ , \quad \tilde L_0^{NS} = \sum_{n\in\mathbb{Z}+1/2} n : \tilde\psi_{-n}^\dagger\tilde\psi_n : -\frac{1}{24} , \tag{2.20b}$$

$$L_0^R = \sum_{n\in\mathbb{Z}} n : \psi_{-n}^\dagger\psi_n : +\frac{1}{12} \ , \quad \tilde L_0^R = \sum_{n\in\mathbb{Z}} n : \tilde\psi_{-n}^\dagger\tilde\psi_n : +\frac{1}{12} , \tag{2.20c}$$

where, in (2.20a), $\beta_0 = p_L$ and $\tilde{\beta}_0 = p_R$. The partition functions for left-movers are

$$\mathrm{Tr}\left[q^{L_0^B}\right] = q^{-1/24}\sum_{p_L} q^{\frac{1}{2}p_L^2}\prod_{n=1}^{\infty}\frac{1}{1-q^n} \tag{2.21a}$$

$$\mathrm{Tr}\left[q^{L_0^{NS}}\right] = q^{-1/24}\prod_{n=0}^{\infty}(1+q^{n+\frac{1}{2}})^2 \tag{2.21b}$$

$$\mathrm{Tr}\left[q^{L_0^R}\right] = q^{1/12}\, 2\prod_{n=1}^{\infty}(1+q^n)^2\,. \tag{2.21c}$$

Using the Jacobi triple product identity

$$\sum_{n\in Z} z^n q^{n^2} = \prod_{n=0}^{\infty}(1-q^{2n+2})(1+zq^{2n+1})(1+z^{-1}q^{2n+1})\,, \tag{2.22}$$

we can derive the identities

$$q^{-\frac{1}{24}}\sum_{n\in\mathbb{Z}} q^{\frac{1}{2}n^2}\prod_{n=1}^{\infty}\frac{1}{1-q^n} = q^{-\frac{1}{24}}\prod_{n=0}^{\infty}(1+q^{n+1/2})^2 \tag{2.23a}$$

$$q^{-\frac{1}{24}}\sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(n+\frac{1}{2})^2}\prod_{n=1}^{\infty}\frac{1}{1-q^n} = q^{\frac{1}{12}}\, 2\prod_{n=1}^{\infty}(1+q^n)^2 \tag{2.23b}$$

$$q^{-\frac{1}{24}}\sum_{n\in\mathbb{Z}}(-1)^n q^{\frac{1}{2}n^2}\prod_{n=1}^{\infty}\frac{1}{1-q^n} = q^{-\frac{1}{24}}\prod_{n=0}^{\infty}(1-q^{n+1/2})^2. \tag{2.23c}$$

Using these identities, we see that boson partition function with p_L restricted in various ways is equivalent to various fermion partition functions:

$$\mathrm{Tr}\left[q^{L_0^B}\right] = \mathrm{Tr}\left[q^{L_0^{NS}}\right] \qquad \text{when } p_L = \frac{1}{2}\cdot(\text{ even integer }) \tag{2.24a}$$

$$\mathrm{Tr}\left[q^{L_0^B}\right] = \mathrm{Tr}\left[q^{L_0^R}\right] \qquad \text{when } p_L = \frac{1}{2}\cdot(\text{ odd integer}) \tag{2.24b}$$

$$\mathrm{Tr}\left[q^{L_0^B}\right] = \mathrm{Tr}\left[q^{L_0^{NS}}\right] + \mathrm{Tr}\left[q^{L_0^R}\right] \qquad \text{when } p_L = \frac{1}{2}\cdot(\text{ integer })\,. \tag{2.24c}$$

A similar consideration holds for the right-moving sector.

It is interesting to ask how we could reproduce the partition function of a compact boson field. Consider the periodic identification

$$\phi \sim \phi + 2\pi R\,. \tag{2.25}$$

Because the coordinate is identified periodically, the total momentum is quantized. In addition, the string world-sheet can wrap the target space cycle, so there is a quantized wrapping number. The quantized total momentum is the sum of left- and right-moving momenta. The wrapping number is proportional to the difference of left- and right-moving momenta, which is therefore also quantized. The result is

$$p_L + p_R = \frac{\sqrt{2}}{R}\cdot\ \text{integer}\,, \qquad p_L - p_R = \sqrt{2}R\cdot\ \text{integer}\,. \tag{2.26}$$

This correlates the momenta in the right and left-handed sectors. The appropriate identifications are

$$R = \frac{1}{\sqrt{2}} \tag{2.27}$$

or its T-dual $R = \sqrt{2}$. Note that both of these would be consistent with the bosonization formulae (2.1a) and (2.1b). (When $R = 1/\sqrt{2}$ the right and left-handed fermions can be double-valued.) We will consider $R = 1/\sqrt{2}$. Then

$$p_L + p_R = 2 \cdot \text{ integer}\,, \qquad p_L - p_R = \text{ integer}\,. \tag{2.28}$$

This determines the spectra of the momentum operators p_L and p_R. Clearly, these equations are solved when both p_L and p_R are integers, or when both are half-odd-integers, with the additional constraint that their sum is an even integer.

To fermionize this system, we must combine different sectors of fermions in such a way as to reproduce this spectrum of momenta. It is straightforward to show that

$$\mathrm{Tr}\left[q^{L_0^B}\bar{q}^{\tilde{L}_0^B}\right] = \mathrm{Tr}\left[\frac{1}{2}\left(1+(-1)^{p_L+p_R}\right) q^{L_0^{NS}}\ \bar{q}^{\tilde{L}_0^{NS}}\right] \ + \ \mathrm{Tr}\left[\frac{1}{2}\left(1+(-1)^{p_L+p_R}\right) q^{L_0^{R}}\ \bar{q}^{\tilde{L}_0^{R}}\right]. \tag{2.29}$$

In the fermion traces, p_L and $-p_R$ are identified with the fermion number operators (2.16). Remember that p_L and p_R are integers in the NS-NS sector and are half-odd-integers in the R-R sector.

In eq. (2.29), we have enforced the condition that $p_L + p_R$ must be an even integer by inserting the projection operator $\frac{1}{2}\left(1+(-1)^{p_L+p_R}\right)$. The trace in the NS-NS sector sums over those states where both p_L and p_R are integers and the sum $p_L + p_R$ is even, therefore the difference $p_L - p_R$ is also even. The trace in the R-R sector sums over states where both p_L and p_R are half-odd-integers, then, when $p_L + p_R$ is even, $p_L - p_R$ is odd. Including both sectors then sums over states where $p_L + p_R$ is even and $p_L - p_R$ are any integers, matching the spectrum of the compact boson with identification $X \sim X + \sqrt{2}\pi$ exactly.

The projection in eq. (2.29) is reminiscent of the GSO projection of the fermionic Neveu-Schwarz-Ramond string which obtains the non-supersymmetric type 0 string theory. In the following, we shall see that different compactifications of closed string coordinates which are consistent with fermionization of those closed string bosonic degrees of freedom generally involve such a projection, and the details of the projection depend on the specific compactification.

2.2 Boundary states

The boundary state is a state of closed string theory which represents the interaction of closed strings with a D-brane. It is a closed string state which is annihilated by the boundary condition that would be imposed on an open string embedding function when the open string world-sheet ends on the D-brane world-volume. Depending on whether the open string embedding function is longitudinal or transverse to the brane, the boundary condition is Neumann or Dirichlet, respectively. Let us assume that the cylindrical world-sheet of the closed string encounters a D-brane when the world-sheet coordinate $\tau = 0$.

JHEP05(2005)072

The Neumann and Dirichlet boundary states for the bosonic string obey[1]

$$\phi_L(0,\sigma)\,|N\rangle = \phi_R(0,\sigma)\,|N\rangle\,, \tag{2.30a}$$

$$\phi_L(0,\sigma)\,|D\rangle = -\phi_R(0,\sigma)\,|D\rangle\,. \tag{2.30b}$$

The conditions (2.30a) and (2.30b) are solved by

$$|N\rangle = \sum_{p_L}\prod_{n=1}^{\infty}\exp\left(-\frac{1}{n}\beta_{-n}\tilde{\beta}_{-n}\right)|p_L,-p_L\rangle\,, \tag{2.31a}$$

$$|D\rangle = \sum_{p_L}\prod_{n=1}^{\infty}\exp\left(\frac{1}{n}\beta_{-n}\tilde{\beta}_{-n}\right)|p_L,p_L\rangle\,. \tag{2.31b}$$

In the fermion representation with eqs. (2.1a) and (2.1b), eqs. (2.30a) and (2.30b) imply that the boundary states satisfy

$$\psi_L(0,\sigma)\ \ |N\rangle = i\psi_R^\dagger(0,\sigma)\,|N\rangle\ \ ,\ \ \psi_L^\dagger(0,\sigma)\,|N\rangle = i\psi_R(0,\sigma)\ \ |N\rangle\,, \tag{2.32a}$$

$$\psi_L(0,\sigma)\ \ |D\rangle = -i\psi_R(0,\sigma)\,|D\rangle\ \ ,\ \ \psi_L^\dagger(0,\sigma)\,|D\rangle = -i\psi_R^\dagger(0,\sigma)\ \ |D\rangle\,. \tag{2.32b}$$

Note that we have chosen cocycles in such a way that they cancel from these boundary equations.

A solution of these equations in the NS-NS sector is

$$|N\rangle_{NS} = \prod_{n=1/2}^{\infty} e^{i\left(\psi^\dagger_{-n}\tilde{\psi}^\dagger_{-n}+\psi_{-n}\tilde{\psi}_{-n}\right)}\,|0\rangle_{NS}\,, \tag{2.33a}$$

$$|D\rangle_{NS} = \prod_{n=1/2}^{\infty} e^{-i\left(\psi_{-n}\tilde{\psi}^\dagger_{-n}+\psi^\dagger_{-n}\tilde{\psi}_{-n}\right)}\,|0\rangle_{NS} \tag{2.33b}$$

and in the RR-sector,

$$|N\rangle_R = \prod_{n=0}^{\infty} e^{i\psi^\dagger_{-n}\tilde{\psi}^\dagger_{-n}}\cdot e^{i\psi_{-n}\tilde{\psi}_{-n})}\,|--\rangle\,, \tag{2.34a}$$

$$|D\rangle_R = \prod_{n=0}^{\infty} e^{-i\psi_{-n}\tilde{\psi}^\dagger_{-n}}\cdot e^{-i\psi^\dagger_{-n}\tilde{\psi}_{-n}}\,|-+\rangle\,. \tag{2.34b}$$

2.3 Open string partition functions

Now that we have constructed boundary states in the fermion representation, it is possible to compare open string partition functions. The boson hamiltonians are given in (2.20a). In the boson theory, the partition function for the open string coordinate with Neumann boundary condition is

$$Z^B_{NN}[q] = \langle N|\,q^{(L_0^B+\tilde{L}_0^B)}\,|N\rangle = \sum_{p_L} q^{p_L^2-\frac{1}{12}}\prod_{n=1}^{\infty}\frac{1}{1-q^{2n}}\,. \tag{2.35}$$

[1]The first of these is equivalent to usual Dirichlet condition $\partial_\tau\phi(0,\sigma)|D\rangle = 0$ plus the additional condition that $(x_L - x_R)|D\rangle = 0$ which we are free to impose.

The momentum p_L here is summed over all integers and half-odd integers. We see this from the following argument: We recall that, with the periodic identification (2.25) of the boson at $R = 1/\sqrt{2}$, the momenta are quantized so that they obey the constraints in eq. (2.28). In the Neuman state, $p_R = -p_L$ comes from the Neuman boundary condition. Therefore $p_L + p_R = 0$ and the first condition, that $p_L + p_R$ be even, is automatic. Further, the second condition, that $p_L - p_R$ must be an integer, tells us that $2p_L$ =integer. Thus, p_L in the summation in (2.35) can take any integer or any half-odd integer value. We will now show that the boson states where p_L is an integer correspond to fermion NS-NS states and those where p_L is half-odd-integer correspond to fermion R-R states.

In the NS-NS sector the hamiltonians are given in (2.20b). The NS-NS contribution to the partition function is

$$\begin{aligned} Z^{NS}_{NN}[q] = & \quad {}_{NS}\langle N|\, q^{(L_0^{NS}+\tilde{L}_0^{NS})}\, |N\rangle_{NS} = q^{-1/12} \prod_{n=1/2}^{\infty} \left(1+q^{2n}\right)^2 \\ & = \sum_{m\in\mathbb{Z}} q^{m^2-1/12} \prod_{n=1}^{\infty} \frac{1}{1-q^{2n}}\,. \end{aligned} \tag{2.36}$$

In the last step, we used the Jacobi triple product identity (2.22) with $z = 1$. Comparing the partition functions, we find that the Neumann state in the NS-NS sector contains only the bosonic states where p_L is an integer.

To get the states where p_L is a half-odd-integer, we must consider the R-R sector. The R-R sector hamiltonians are given in (2.20c) and the R-R contribution to the partition function is

$$\begin{aligned} Z^{R}_{NN}[q] = & \quad {}_R\langle N|q^{\left(L_0^R+\tilde{L}_0^R\right)}|N\rangle_R \\ & = 2q^{\frac{1}{6}} \prod_{n=1} (1+q^{2n})^2 = \sum_{m} q^{\left(\frac{2m+1}{2}\right)^2-\frac{1}{12}} \prod_{n=1}^{\infty} \frac{1}{(1-q^{2n})}\,. \end{aligned} \tag{2.37}$$

Indeed we see that, the R-R sector of the fermion theory corresponds to the boson states where p_L is a half-odd-integer.

Then, in order that the fermion theory produce the correct partition function, we need both the NS-NS and R-R sectors,

$$Z^{B}_{NN}[q] = Z^{NS}_{NN}[q] \;+\; Z^{R}_{NN}[q]\,. \tag{2.38}$$

Here, we could have completed the projection analogous to eq. (2.29) by including the projector $\frac{1}{2}(1+(-1)^{p_L+p_R})$ in the traces. However, in both the NS-NS and R-R fermion Neumann boundary states, the total p_L+p_R, which is the difference of right and left-handed fermion numbers, is zero. Thus, the projection would have trivial effect and we have not done it explicitly.

In summary, the description of the open string coordinate with Neumann boundary conditions in fermion variables requires use of both the NS-NS and the R-R sectors and the projection onto states with even total momentum, which is trivial in this case.

JHEP05(2005)072

The boson partition function for Dirichlet states is

$$\begin{aligned} Z^B_{DD}[q] &= \langle D| q^{\left(L^B_0+\tilde{L}^B_0\right)} |D\rangle \\ &= \sum_{p_L} q^{\left[p_L^2-\frac{1}{12}\right]} \prod_{n=1} \frac{1}{1-q^{2n}}\,. \end{aligned} \tag{2.39}$$

Now, since $p_L + p_R$ must be an even integer and $p_L = p_R$, p_L must be an integer. There are no half-odd-integer momentum states in this case.

By some simple algebra we find the partition function $Z_{DD}[q]$ in the fermion theory, using the fermion boundary states $|D\rangle$ eq. (2.33b) and eq. (2.34b). In the NS-NS sector

$$\begin{aligned} Z^{NS}_{DD}[q] &= \langle D|\frac{1}{2}\left(1+(-1)^{p_L+p_R}\right) q^{\left(L^N_0+\tilde{L}^{NS}_0\right)}|D\rangle \\ &= q^{-\frac{1}{12}} \prod_{n=\frac{1}{2}} (1+q^{2n})^2 = \sum_{m\in\mathbb{Z}} q^{\left[m^2-\frac{1}{12}\right]} \prod_{n=1} \frac{1}{1-q^{2n}}\,. \end{aligned} \tag{2.40}$$

Note that $p_L + p_R$ is an even integer in all NS-NS Dirichlet states so the projection is a trivial operation here. However, because of fermion zero modes, it is odd in R Dirichlet states, so the R-R sector partition function must vanish,

$$Z^R_{DD}[q] = \langle D|\frac{1}{2}\left(1+(-1)^{p_L+p_R}\right) q^{\left(L^R_0+\tilde{L}^R_0\right)}|D\rangle = 0\,. \tag{2.41}$$

We see again that the projected combination of fermion NS-NS and R-R sectors indeed produces the boson partition function for open strings.

3. Rolling tachyon and fermionization

After the Wick rotation, the rolling tachyon is described by the boundary conformal field theory (We choose $B = 0$ and $A = \frac{g}{2}$.)

$$S = \frac{1}{4\pi}\int d\tau d\sigma \partial_a X \partial_a X - ig\pi \oint \frac{d\sigma}{2\pi} e^{iX}\,. \tag{3.1}$$

Note that the factor of i in front of the interaction comes from the fact that the world-sheet coordinates are also Wick rotated to Euclidean space and that the boundary integration variable σ is a spatial coordinate here, so is not Wick rotated.

It is this theory which is solvable and which we shall represent using fermions. We will do this by constructing a boundary state. It is a state of closed string theory which is annihilated by the open string theory boundary condition,

$$\left.\left(\frac{1}{2\pi}\frac{\partial}{\partial\tau}X(\tau,\sigma) + \frac{g}{2} : e^{iX(\tau,\sigma)} :\right)\right|_{\tau=0} |B\rangle = 0. \tag{3.2}$$

This is the boundary condition that one would impose in order to avoid surface contributions to the equation of motion when it is derived from varying the action (3.1). Our goal is to find the boundary state $|\mathcal{B}\rangle$ which satisfies this equation.

There is an important obstacle to introducing fermions in the most straightforward way. The periodic potential for the boson $X(\tau,\sigma)$ in (3.1) has the wrong period to produce the fermionized closed string degrees of freedom which we discussed in the last section. Recall that, in order to have a sensible fermion representation, the boson had to be identified with period $\phi(\tau,\sigma) \sim \phi(\tau,\sigma)+\sqrt{2}\pi$. Here the period is $X(\tau,\sigma) \sim X(\tau,\sigma)+2\pi$.

Following ref. [40], we observe that this problem can be fixed by doubling the boson degrees of freedom. Accordingly, we introduce a free boson $Y(\tau,\sigma)$ such that the action (3.1) is replaced by

$$S=\frac{1}{4\pi}\int d\tau d\sigma\,(\partial_a X\partial_a X+\partial_a Y\partial_a Y)-ig\pi\int\frac{d\sigma}{2\pi}e^{iX(0,\sigma)}\,. \tag{3.3}$$

We will assume that the Y-boson has a Dirichlet boundary condition and that, like X, it is periodically identified with

$$Y\sim Y+2\pi\,.$$

We will then construct a double boundary state, $|B,D\rangle$ which obeys the boundary condition for X as in eq. (3.2) as well as an additional condition for Y,

$$\left(\frac{1}{2\pi}\frac{\partial}{\partial\tau}X(0,\sigma)+\frac{g}{2}:e^{iX(0,\sigma)}:\right)\;|B,D\rangle\,=0\,, \tag{3.4a}$$

$$Y(0,\sigma)\;|B,D\rangle\,=0\,. \tag{3.4b}$$

Since X and Y do not interact, it will be always be easy to separate the contribution of the Y-field from matrix elements. Its contribution factors into one for a free boson with a Dirichlet boundary condition which is easily identified and discarded in the amplitudes that we compute.

The Dirichlet boundary condition for Y in (3.4b) implies

$$[\partial_\tau Y_L(0,\sigma)-\partial_\tau Y_R(0,\sigma)]\,|B,D\rangle=0\,. \tag{3.5}$$

Using (3.4b) and (3.5), we can re-write (3.4a) as

$$\begin{aligned}\Bigg[\frac{1}{2\pi}\frac{\partial}{\partial\tau}\,(X_L(0,\sigma)+Y_L(0,\sigma))+\frac{1}{2\pi}\frac{\partial}{\partial\tau}\,(X_R(0,\sigma)-Y_R(0,\sigma))\,+\\ +\,\frac{g}{2}:e^{i(X_L(0,\sigma)+Y_L(0,\sigma))}:\;:e^{i(X_R(0,\sigma)+Y_R(0,\sigma))}:\Bigg]\;|B,D\rangle=0\,.\end{aligned} \tag{3.6}$$

An infinite factor from normal ordering $e^{iY(0,\sigma)}=e^{iY_L(0,\sigma)}e^{iY_R(0,\sigma)}$ has been absorbed into g. We can re-express the boson fields as the canonically normalized pair

$$\phi_{1L}(\tau,\sigma)=\frac{X_L(\tau,\sigma)+Y_L(\tau,\sigma)}{\sqrt{2}}\,,\qquad \phi_{1R}(\tau,\sigma)=\frac{X_R(\tau,\sigma)+Y_R(\tau,\sigma)}{\sqrt{2}} \tag{3.7a}$$

$$\phi_{2L}(\tau,\sigma)=\frac{X_L(\tau,\sigma)-Y_L(\tau,\sigma)}{\sqrt{2}}\,,\qquad \phi_{2R}(\tau,\sigma)=\frac{X_R(\tau,\sigma)-Y_R(\tau,\sigma)}{\sqrt{2}}\,. \tag{3.7b}$$

In terms of these, we obtain

$$\left(\partial_\tau\phi_{1L}(0,\sigma)+\partial_\tau\phi_{2R}(0,\sigma)+\frac{g\pi}{\sqrt{2}}:e^{\sqrt{2}i\phi_{1L}(0,\sigma)}:\ \ :e^{\sqrt{2}i\phi_{1R}(0,\sigma)}:\right)\ |B,D\rangle=0\,, \tag{3.8a}$$

$$[\phi_{1L}(0,\sigma)+\phi_{1R}(0,\sigma)-\phi_{2L}(0,\sigma)-\phi_{2R}(0,\sigma)]\ |B,D\rangle=0\,. \tag{3.8b}$$

Note that, normal ordering for (X,Y) implies normal ordering for (ϕ_1,ϕ_2).

The problem that has been solved here is the following: Recalling our discussion of fermionizing a compact boson surrounding eqs. (2.26) and (2.27), we see that, when written in terms of the variables ϕ_1 and ϕ_2, the equation for the boundary state (3.8a), has the correct periodicity for identification of the exponentials with fermion operators. This is in contrast with its previous form (3.2) in terms of X .The definitions (3.7a) and (3.7b) of ϕ_1 and ϕ_2, imply the identifications $\phi_1\sim\phi_1+\sqrt{2}\pi$ and $\phi_2\sim\phi_2+\sqrt{2}\pi$ when $X\sim X+2\pi$ and $Y\sim Y+2\pi$.

3.1 Fermion representation of the doubled bosons

We will now need to generalize fermionization to the doubled system. There are two species of left and right-handed fermions, together with their conjugates,

$$\psi_{1L}(z)=\eta_{1L}\ \ :e^{-i\sqrt{2}\phi_{1L}(z)}:,\psi_{1R}(\bar z)=\eta_{1R}\ \ :e^{i\sqrt{2}\phi_{1R}(\bar z)}:, \tag{3.9a}$$

$$\psi_{2L}(z)=\eta_{2L}\ \ :e^{i\sqrt{2}\phi_{2L}(z)}:,\psi_{2R}(\bar z)=\eta_{2R}\ \ :e^{-i\sqrt{2}\phi_{2R}(\bar z)}:. \tag{3.9b}$$

The explicit expression of the cocycles is given in the appendix. The cocycles have been carefully adjusted so that all independent fields anti-commute. Some further freedom is used to adjust them so that they do not appear explicitly in other formulae, such as the fermion boundary conditions that we will find below. The fermion currents are given by

$$i\sqrt{2}\partial_\tau\phi_{1L}(\tau,\sigma)=\,:\psi^\dagger_{1L}(\tau,\sigma)\psi_{1L}(\tau,\sigma):,\ \ -i\sqrt{2}\partial_\tau\phi_{1R}(\tau,\sigma)=:\psi^\dagger_{1R}(\tau,\sigma)\psi_{1R}(\tau,\sigma): \tag{3.10a}$$

$$-i\sqrt{2}\partial_\tau\phi_{2L}(\tau,\sigma)=\,:\psi^\dagger_{2L}(\tau,\sigma)\psi_{2L}(\tau,\sigma):,\ i\sqrt{2}\partial_\tau\phi_{2R}(\tau,\sigma)=:\psi^\dagger_{2R}(\tau,\sigma)\psi_{2R}(\tau,\sigma):. \tag{3.10b}$$

In order to construct the rolling tachyon boundary state we shall need a fermion representation of $|N,D\rangle$. In boson variables it obeys the boundary condition

$$(X_L(0,\sigma)-X_R(0,\sigma))\,|N,D\rangle=0\,,\qquad (Y_L(0,\sigma)+Y_R(0,\sigma))\,|N,D\rangle=0 \tag{3.11}$$

or

$$\phi_{1L}(0,\sigma)\,|N,D\rangle=\phi_{2R}(0,\sigma)\,|N,D\rangle=0\,,\qquad \phi_{2L}(0,\sigma)\,|N,D\rangle=\phi_{1R}(0,\sigma)\,|N,D\rangle\,. \tag{3.12}$$

In fermion variables,

$$\psi_{1L}(0,\sigma)|N,D\rangle=\psi_{2R}(0,\sigma)|N,D\rangle,\quad \psi_{2L}(0,\sigma)|N,D\rangle=-\psi_{1R}(0,\sigma)|N,D\rangle\,, \tag{3.13a}$$

$$\psi^\dagger_{1L}(0,\sigma)|N,D\rangle=-\psi^\dagger_{2R}(0,\sigma)|N,D\rangle\,,\qquad \psi^\dagger_{2L}(0,\sigma)|N,D\rangle=\psi^\dagger_{1R}(0,\sigma)|N,D\rangle\,. \tag{3.13b}$$

We will construct this state (and others) explicitly in the next section. Note that eq. (3.12) already implies the boundary condition in eq. (3.8b).

Now, eq. (3.8a) becomes

$$\Big[: \psi^\dagger_{1L}(0,\sigma)\psi_{1L}(0,\sigma) : + : \psi^\dagger_{2R}(0,\sigma)\psi_{2R}(0,\sigma) : + ig\pi : \psi^\dagger_{1L}(0,\sigma)\psi_{1R}(0,\sigma) :\Big] |B, D\rangle = 0\,. \tag{3.14}$$

A solution of this equation is

$$|B, D\rangle = \exp\left(ig\pi \int \frac{d\sigma}{2\pi} : \psi^\dagger_{1L}(0,\sigma)\psi_{2L}(0,\sigma) :\right) \; |N, D\rangle\;. \tag{3.15}$$

To see this, note that, substituting (3.15) into (3.14) implies

$$\begin{aligned}&\Big[: \psi^\dagger_{1L}(0,\sigma)\psi_{1L}(0,\sigma) : + : \psi^\dagger_{2R}(0,\sigma)\psi_{2R}(0,\sigma) : - \\ &+ ig\pi\psi^\dagger_{1L}(0,\sigma)\left(\psi_{1R}(0,\sigma) + \psi_{2L}(0,\sigma)\right)\Big] \; |N, D\rangle = 0\,.\end{aligned} \tag{3.16}$$

Indeed, using eqs. (3.13a) and (3.13b) we see that this is an identity as both the combination of the first two terms and the last term annihilate the state $|N, D\rangle$.

In the next section, we will give an explicit construction of the Neumann-Dirichlet state $|N, D\rangle$. Then, combining the result with (3.15) will finally yield the complete solution of the problem of finding a fermion representation of the rolling tachyon boundary state.

A further observation is that the fermion boundary condition for the rolling tachyon boundary state can be presented as

$$\left(\psi_{1L}(0,\sigma) - i\pi g\psi_{2L}(0,\sigma)\right)|B, D\rangle = \psi_{2R}(0,\sigma)|B, D\rangle\,, \tag{3.17a}$$

$$-\psi_{2L}(0,\sigma)|B, D\rangle = \psi_{1R}(0,\sigma)|B, D\rangle\,, \tag{3.17b}$$

$$\left(\psi^\dagger_{2L}(0,\sigma) + i\pi g\psi^\dagger_{1L}(0,\sigma)\right)|B, D\rangle = \psi^\dagger_{1R}(0,\sigma)|B, D\rangle\,, \tag{3.17c}$$

$$-\psi^\dagger_{1L}(0,\sigma)|B, D\rangle = \psi^\dagger_{2R}(0,\sigma)|B, D\rangle\,. \tag{3.17d}$$

3.2 Quantum theory of the bosons (ϕ_1, ϕ_2)

To fix our conventions, we shall examine the free boson fields $(\phi_1, \phi_2) = (\frac{1}{\sqrt{2}}(X+Y), \frac{1}{\sqrt{2}}(X-Y))$ with action

$$S = \frac{1}{4\pi}\int d\tau d\sigma\left((\partial\phi_1)^2 + (\partial\phi_2)^2\right) = \frac{1}{4\pi}\int d\tau d\sigma\left((\partial X)^2 + (\partial Y)^2\right). \tag{3.18}$$

They have the mode expansion

$$\phi_{aL}(\tau + i\sigma) = \frac{1}{\sqrt{2}}\chi_{aL} - \frac{i}{\sqrt{2}}\pi_{aL}(\tau + i\sigma) + \frac{i}{\sqrt{2}}\sum_{n\neq 0}\frac{1}{n}\beta_{an}e^{-n(\tau+i\sigma)}\,, \tag{3.19a}$$

$$\phi_{aR}(\tau - i\sigma) = \frac{1}{\sqrt{2}}\chi_{aR} - \frac{i}{\sqrt{2}}\pi_{aR}(\tau - i\sigma) + \frac{i}{\sqrt{2}}\sum_{n\neq 0}\frac{1}{n}\tilde{\beta}_{an}e^{-n(\tau-i\sigma)}\,. \tag{3.19b}$$

The non-vanishing commutation relations are

$$[\chi_{aL}, \pi_{bL}] = i\delta_{ab}, \quad [\chi_{aR}, \pi_{bR}] = i\delta_{ab}\,,$$
$$[\beta_{an}, \beta_{bm}] = n\delta(n+m)\delta_{ab}, \quad \left[\tilde\beta_{an}, \tilde\beta_{bm}\right] = n\delta(n+m)\delta_{ab}\,. \tag{3.20}$$

The doubled fermions have mode expansions

$$\psi_{aL}(\tau+i\sigma) = \sum_n \psi_{an} e^{-n(\tau+i\sigma)}\,, \qquad \psi^\dagger_{aL}(\tau+i\sigma) = \sum_n \psi^\dagger_{an} e^{-n(\tau+i\sigma)} \tag{3.21a}$$
$$\psi_{aR}(\tau-i\sigma) = \sum_n \tilde\psi_{an} e^{-n(\tau-i\sigma)}\,, \qquad \psi^\dagger_{aR}(\tau-i\sigma) = \sum_n \tilde\psi^\dagger_{an} e^{-n(\tau-i\sigma)}\,, \tag{3.21b}$$

where the non-vanishing anti-commutators are

$$\left\{\psi_{am}, \psi^\dagger_{bn}\right\} = \delta_{ab}\delta_{m+n}\,, \qquad \left\{\tilde\psi_{am}, \tilde\psi^\dagger_{bn}\right\} = \delta_{ab}\delta_{m+n}\,. \tag{3.22}$$

In our discussion of a fermion representation of a single boson, we noted that the momenta (π_{1L}, π_{1R}) are identified with fermion numbers, and they also control the periodicity of the fermions under $\sigma \to \sigma + 2\pi$. This quantization of the fermion number and therefore the momenta requires a periodic identification of the boson. For example, if $\phi_{1L} \sim \phi_{1L} + \sqrt{2}\pi$, then p_{1L} is quantized either as integers or half-odd-integers. Integer momentum corresponds an NS fermion whereas half-odd-integer momentum corresponds to an R fermion. Then, for example, to get all of the states of the compact boson $\phi_1 = \phi_{1L} + \phi_{1R}$ with $\phi_1 \sim \phi_1 + \sqrt{2}\pi$ we must take the NS-NS and R-R states of the fermions (ψ_{1L}, ψ_{1R}) and further project onto states where $\pi_1 = \pi_{1L} + \pi_{1R}$ is an even integer.

A similar construction obtains all of the states of the compact boson ϕ_2 and the total space of all states is the direct product of these representations. However, these are not quite the correct states to describe the theory which has compact bosons X and Y with identifications $X \sim X + 2\pi$ and $Y \sim Y + 2\pi$. To correctly describe them, we need the states where the momenta of X and Y, rather than ϕ_1 and ϕ_2 are quantized appropriately. It is easy to see that these are the states where

$$\pi_{1L} + \pi_{2L} + \pi_{1R} + \pi_{2R} = 2m_X \tag{3.23a}$$
$$\pi_{1L} - \pi_{2L} + \pi_{1R} - \pi_{2R} = 2m_Y \tag{3.23b}$$
$$\pi_{1L} + \pi_{2L} - \pi_{1R} - \pi_{2R} = 2w_X \tag{3.23c}$$
$$\pi_{1L} - \pi_{2L} - \pi_{1R} + \pi_{2R} = 2w_Y\,, \tag{3.23d}$$

where (m_X, m_Y, w_X, w_Y) are integers. In order to satisfy eqs. (3.23a)–(3.23d), there are two possibilities: all momenta $(\pi_{1L}, \pi_{1R}, \pi_{2L}, \pi_{2R})$ must be integers, or all momenta $(\pi_{1L}, \pi_{1R}, \pi_{2L}, \pi_{2R})$ must be half-odd integers. As a result, both fermions (ψ_{1L}, ψ_{1R}) and (ψ_{2L}, ψ_{2R}) must be NS-NS, or both must be R-R. Then, in addition to this constraint, to obtain all states satisfying satisfy eqs. (3.23a)–(3.23d), it is sufficient to project onto states where one of the combinations in (3.23a)-(3.23d) is even, for example states where $\pi_{1L} - \pi_{2L} - \pi_{1R} + \pi_{2R}$ is even. This can be done with the projection operator

$$P = \frac{1}{2}\left[1 + (-1)^{\pi_{1L} - \pi_{1R} - \pi_{2L} + \pi_{2R}}\right]. \tag{3.24}$$

These are the states where, in our convention for identifying momenta and fermion numbers, the total fermion number is even.

JHEP05(2005)072

To confirm that this is the correct projection of the fermion states, let us compare partition functions. The partition function for the X-boson is

$$\begin{aligned}
Z[q,\bar{q}] &= \mathrm{Tr}\left[q^{L_{X0}}\bar{q}^{\tilde{L}_{X0}}\right] \\
&= \sum_{p_{XL},p_{XR}} q^{\frac{1}{2}p_{XL}^2}\bar{q}^{\frac{1}{2}p_{XR}^2}\cdot\left|q^{-\frac{1}{24}}\prod_1^\infty\frac{1}{1-q^n}\right|^2 \\
&= \left[\sum_{m,n\in\mathbb{Z}} q^{\frac{1}{4}(m+n)^2}\bar{q}^{\frac{1}{4}(m-n)^2}\right]\cdot\left|q^{-\frac{1}{24}}\prod_1^\infty\frac{1}{1-q^n}\right|^2 .
\end{aligned}\tag{3.25}$$

Since both m and n are integers, $m+n$ and $m-n$ must either both be even or both odd. Thus,

$$Z[q,\bar{q}] = \left[\left|\sum_{n\in\mathbb{Z}} q^{n^2}\right|^2 + \left|\sum_{n\in\mathbb{Z}} q^{(n+\frac{1}{2})^2}\right|^2\right]\cdot\left|q^{-\frac{1}{24}}\prod_1^\infty\frac{1}{1-q^n}\right|^2 \tag{3.26}$$

and the partition function for the (X,Y) system is just the square, $(Z[q,\bar{q}])^2$.

We will now see if we recover this result in the fermion theory. In the projected NS-NS sector we get

$$Z_{NS}[q,\bar{q}] = \mathrm{Tr}\left[Pq^{L_0^{NS}}\bar{q}^{\tilde{L}_0^{NS}}\right] = \frac{1}{2}\left|q^{-\frac{1}{24}}\prod_0^\infty(1+q^{n+\frac{1}{2}})\right|^4 + \frac{1}{2}\left|q^{-\frac{1}{24}}\prod_0^\infty(1-q^{n+\frac{1}{2}})\right|^4 . \tag{3.27}$$

Here, the first term comes from the 1 in the projection operator. The second term comes from the second term in the projection operator, which changes the sign of the excites states. The Jacobi identity leads to

$$Z_{NS}[q,\bar{q}] = \frac{1}{2}\left(\left|\sum_{n\in\mathcal{Z}} q^{\frac{1}{2}n^2}\right|^4 + \left|\sum_{n\in\mathcal{Z}}(-1)^n q^{\frac{1}{2}n^2}\right|^4\right)\left|q^{-\frac{1}{24}}\prod_{n=1}^\infty\frac{1}{1-q^n}\right|^4 \tag{3.28}$$

and in the projected R-R sector,

$$\begin{aligned}
Z_R[q,\bar{q}] &= \mathrm{Tr}\left[Pq^{L_0^R}\bar{q}^{\tilde{L}_0^R}\right] = \frac{1}{2}\left|q^{\frac{1}{12}}2\prod_1^\infty(1+q^n)\right|^4 \\
&= \frac{1}{2}\left|\sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(n+\frac{1}{2})^2}\right|^4\left|q^{-\frac{1}{24}}\prod_{n=1}^\infty\frac{1}{1-q^n}\right|^4 .
\end{aligned}\tag{3.29}$$

Here, we have used the identities in eqs. (2.23a)–(2.23c). Indeed, by comparing the partition functions,

$$\begin{aligned}
(Z[q,\bar{q}])^2 &= Z_{NS}[q,\bar{q}] + Z_R[q,\bar{q}] \\
&= \left(1+8|q|^{\frac{1}{2}}+16|q|+4q+4\bar{q}+16q^{\frac{1}{4}}\bar{q}^{\frac{5}{4}}+16q^{\frac{5}{4}}\bar{q}^{\frac{1}{4}}+4q^2+4\bar{q}^2+16q\bar{q}+\dots\right)\times \\
&\quad\times\left|q^{-\frac{1}{24}}\prod_{n=1}^\infty\frac{1}{1-q^n}\right|^4 \\
&= \left(1+4|q|^{\frac{1}{2}}+2q+2\bar{q}+\dots\right)^2\left|q^{-\frac{1}{24}}\prod_{n=1}^\infty\frac{1}{1-q^n}\right|^4 .
\end{aligned}\tag{3.30}$$

3.3 Current algebra

It is well known that the vertex operator $: e^{2iX_L(z)} :$, together with $: e^{-2iX_L(z)} :$ and $i\partial_\tau X(z)$ form a level-one SU(2) Kac-Moody algebra when the theory is defined on a circle with the self-dual radius $R_X = 1$

$$\begin{aligned}
J^+(z) &= : e^{2iX_L(z)} :, \quad J^-(z) =: e^{-2iX_L(z)} :, \quad J^3(z) = iz\frac{\partial X_L}{\partial z}(z)\,, \\
J^a(z) &= \sum_n J^a_n \frac{1}{z^n} \\
\left[J^3_n, J^3_m\right] &= \frac{n}{2}\delta_{n+m}\,, \qquad \left[J^+_n, J^-_m\right] = 2J^3_{n+m} + n\delta_{n+m}\,, \\
\left[J^3_n, J^+_m\right] &= J^+_{n+m}\,, \qquad \left[J^3_n, J^-_m\right] = -J^-_{n+m}\,.
\end{aligned} \tag{3.31}$$

This SU(2) algebra plays an important role when one constructs the boundary states for the conformal field theory with the periodic boundary interaction. It is known [4, 5] that the boundary state can be written in terms of the Ishibashi states as

$$|B\rangle = \sum_j \sum_{m\geq 0} \begin{pmatrix} j+m \\ 2m \end{pmatrix} (i\pi g)^{2m} |j,m,m\rangle\rangle, \tag{3.32}$$

where $|j,m,m\rangle\rangle$ are the SU(2) Ishibashi states and $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$.

However, one drawback of this approach is that it is quite laborious to trace the explicit time dependence of the Ishibashi states beyond the first few levels. Furthermore, it is almost impossible to work out the explicit time dependence of all Ishibashi states. Because of this difficulty, a complete description of the long time behavior of the full boundary state is still lacking. (See, however, recent discussion on this point by Sen [45].)

For practical purposes, it would often be more convenient to rewrite the boundary state in the closed string oscillator basis. In particular, it would be useful for the study of its physical properties such as time evolution and closed string emission. However, it is quite difficult to express the Ishibashi states in general in such a way. An exception is the scalar sector which does not involve any oscillators. If we expand the boundary state in the bosonic oscillator basis, we obtain

$$\begin{aligned}
|B\rangle &= f(t)|0,t\rangle + g(t)\alpha_{-1}\tilde{\alpha}_{-1}|0,t\rangle + \ldots\,, \\
f(t) &= \sum_j (i\pi g)^{2j} \langle 0,t|j;j,j\rangle\,.
\end{aligned} \tag{3.33}$$

Adopting the phase convention for $|j;j,j\rangle$ as in [1], we have

$$f(t) = \sum_j (-\pi g e^t)^{2j} = \frac{1}{1+\pi g e^t}\,, \tag{3.34}$$

where we have done the reverse Wick rotation back to lorentzian time.

In the fermion theory, the degrees of freedom are doubled. Indeed, there are two species of left-moving fermions ψ_{1L} and ψ_{2L} which we expect to form a doublet under $\mathrm{SU}(2)_L$. This $\mathrm{SU}(2)_L$ Kac-Moody algebra is realized as the current algebra of the fermion currents

$$J^+_L(z) = : \psi^\dagger_{1L}(z)\psi_{2L}(z) :=: \Psi^\dagger_L(z)\frac{(\sigma_1 + i\sigma_2)}{2}\Psi_L(z) :,$$

$$
\begin{aligned}
J_L^-(z) &= :\psi_{2L}^\dagger(z)\psi_{1L}(z) :=: \Psi_L^\dagger(z)\frac{(\sigma_1 - i\sigma_2)}{2}\Psi_L(z) : \\
J_L^3(z) &= \frac{1}{2}\Big(:\psi_{1L}^\dagger(z)\psi_{1L}(z) : - :\psi_{2L}^\dagger(z)\psi_{2L}(z) : \Big) =: \Psi_L^\dagger(z)\frac{\sigma_3}{2}\Psi_L(z) :
\end{aligned} \tag{3.35}
$$

where

$$
\Psi_L(z) = \begin{pmatrix} \psi_{1L}(z) \\ \psi_{2L}(z) \end{pmatrix}, \qquad \Psi_R(\bar{z}) = \begin{pmatrix} \psi_{1R}(\bar{z}) \\ \psi_{2R}(\bar{z}) \end{pmatrix}.
$$

We might expect that the doubling of degrees of freedom comes with an enhanced symmetry. Indeed, if we decompose the complex fermions into real and imaginary parts, we find an $\mathrm{SO}(4)_L = \mathrm{SU}(2)_L \times \mathrm{SU}(2)'_L$ algebra. In terms of the complex generators, the $\mathrm{SU}(2)_L$ algebra is (3.35) and $\mathrm{SU}(2)'_L$ is

$$
\begin{aligned}
J'^{+}_L(z) &= :\psi_{1L}^\dagger(z)\psi_{2L}^\dagger(z) := i : \Psi_L^\dagger(z)\frac{\sigma^2}{2}\Psi_L^{\dagger t}(z) :, \\
J'^{-}_L(z) &= :\psi_{2L}(z)\psi_{1L}(z) := -i : \Psi_L^t(z)\frac{\sigma^2}{2}\Psi_L(z) : \qquad (3.36) \\
J'^{3}_L(z) &= \frac{1}{2}\Big(:\psi_{1L}^\dagger(z)\psi_{1L}(z) : + :\psi_{2L}^\dagger(z)\psi_{2L}(z) : \Big) = \frac{1}{2} : \Psi_L^\dagger(z)\Psi_L(z) : \qquad (3.37)
\end{aligned}
$$

The algebras (3.35) and (3.37) commute with each other and each is a level-1 SU(2) Kac-Moody algebra.

Note that, because ψ_{1L} and ψ_{2L} have the same boundary condition in the NS-NS and R-R sectors, both sets of charges J_0^a and J'^a_0 commute with the hamiltonian and the symmetries are not broken by boundary conditions. A similar discussion applies to right-handed fermions.

4. Exact boundary state for the unstable D-brane

In the section 2 we studied the fermion representation of a conformal field theory with a single boson. The Neumann and Dirichlet boundary conditions in the boson theory were shown to be realized as linear boundary conditions in the fermion theory and the boundary states were constructed explicitly. In section 3 we constructed the boundary state of the rolling tachyon as a certain global chiral SU(2) rotation of a mixed Neumann-Dirichlet state of the X and Y bosons. It remains to complete our task of finding the rolling tachyon boundary state by finding the explicit Neumann-Dirichlet boundary state in the fermion theory. We will address this problem in this section. The result will be a fermionic representation of the exact boundary state for the unstable D-brane. To begin, in the next subsection, we will construct some simple boundary states with combinations of Neumann and Dirichlet boundary conditions for X and Y.

4.1 Some simple boundary states

In this subsection, we will consider mixed boundary states $|N,N\rangle$, $|N,D\rangle$, $|D,N\rangle$ and $|D,D\rangle$ where N and D denote Neumann and Dirichlet boundary conditions and the first and second label is the condition for the X and Y bosons, respectively. We will translate the boundary conditions to conditions on the fermion variables. We then use these conditions to construct the boundary states in the fermion representation. In order to get all of the states of the boson theory, this must be done in both the NS-NS and R-R sectors of the fermion theory. We will also show that these boundary states are related to each other by simple global $\mathrm{SU}(2)_L \otimes \mathrm{SU}(2)'_L$ rotations.

We begin with the $|N,N\rangle$ state which obeys the boundary conditions

$$X_L(0,\sigma)|N,N\rangle = X_R(0,\sigma)|N,N\rangle, \quad Y_L(0,\sigma)|N,N\rangle = Y_R(0,\sigma)|N,N\rangle\,, \tag{4.1}$$

This implies the fermion boundary conditions (see the appendix)

$$(\psi_{aL}(0,\sigma) - i\psi^\dagger_{aR}(0,\sigma))|N,N\rangle = 0, \quad (\psi^\dagger_{aL}(0,\sigma) - i\psi_{aR}(0,\sigma))|N,N\rangle = 0\,, \qquad a = 1,2\,. \tag{4.2}$$

The boundary state which obeys these conditions is, in the NS-NS sector,

$$\begin{aligned}|N,N\rangle_{NS} = \prod_{r=\frac{1}{2}}^{\infty} &\left(1 + i\psi^\dagger_{1,-r}\tilde{\psi}^\dagger_{1,-r}\right)\left(1 + i\psi_{1,-r}\tilde{\psi}_{1,-r}\right)\left(1 + i\psi^\dagger_{2,-r}\tilde{\psi}^\dagger_{2,-r}\right) \times \\ &\times \left(1 + i\psi_{2,-r}\tilde{\psi}_{2,-r}\right)\ |0\rangle\end{aligned} \tag{4.3}$$

and in the R-R sector

$$\begin{aligned}|N,N\rangle_{R} = \prod_{n=0}^{\infty} &\left(1 + i\psi^\dagger_{1,-n}\tilde{\psi}^\dagger_{1,-n}\right)\left(1 + i\psi_{1,-n}\tilde{\psi}_{1,-n}\right)\left(1 + i\psi^\dagger_{2,-n}\tilde{\psi}^\dagger_{2,-n}\right) \times \\ &\times \left(1 + i\psi_{2,-n}\tilde{\psi}_{2,-n}\right)|----\rangle\,.\end{aligned} \tag{4.4}$$

Here, we use the notation $|----\rangle = |--\rangle|--\rangle$ where the ordering of zero mode state labels is $\psi_{1L}, \psi_{1R}, \psi_{2L}, \psi_{2R}$. Note that the oscillator parts of the boundary states in the two sectors are practically identical. A difference lies in the vacuum states. In the NS-NS sector we use the direct product of the NS-NS vacua for each of the two species of fermions. In the R-R sector, the vacuum is degenerate and we must choose the combination of states which obeys the boundary condition.

Note that the boundary conditions only determine the boundary states up to a phase. In the NS-NS sector, the phase has been fixed by requiring that $\langle 0|N,N\rangle_{NS} = 1$, which matches the phase convention which we choose in the boson theory. We will do this consistently in the NS-NS sector for all of the boundary states that we construct. In the RR-sector we shall discuss a similar matching of the phases in a later section. That matching of phases is anticipated in the formulae for R sector states presented in this section.

The boundary state $|N,D\rangle$ has the boundary condition

$$X_L(0,\sigma)|N,D\rangle = X_R(0,\sigma)|N,D\rangle\,, \qquad Y_L(0,\sigma)|N,D\rangle = -Y_R(0,\sigma)|N,D\rangle\,. \tag{4.5}$$

In terms of fermions (see appendix),

$$(\psi_{1L}(0,\sigma) - \psi_{2R}(0,\sigma))|N,D\rangle = 0\,, \qquad (\psi_{2L}(0,\sigma) + \psi_{1R}(0,\sigma))|N,D\rangle = 0\,, \tag{4.6}$$

$$(\psi^\dagger_{1L}(0,\sigma) + \psi^\dagger_{2R}(0,\sigma))|N,D\rangle = 0\,, \qquad (\psi^\dagger_{2L}(0,\sigma) - \psi^\dagger_{1R}(0,\sigma))|N\,,D\rangle = 0\,. \tag{4.7}$$

The boundary state $|N,D\rangle$ is

$$
\begin{aligned}
|N,D\rangle_{NS} \,=\, \prod_{r=\frac{1}{2}}^{\infty} & \left(1+\tilde{\psi}_{1,-r}\psi^{\dagger}_{2,-r}\right)\left(1-\tilde{\psi}^{\dagger}_{1,-r}\psi_{2,-r}\right)\left(1+\psi^{\dagger}_{1,-r}\tilde{\psi}_{2,-r}\right)\times \\
& \times\left(1-\psi_{1,-r}\tilde{\psi}^{\dagger}_{2,-r}\right)|0\rangle
\end{aligned}
\tag{4.8}
$$

$$
\begin{aligned}
|N,D\rangle_{R} \,=\, -i\prod_{n=0}^{\infty} & \left(1+\tilde{\psi}_{1,-n}\psi^{\dagger}_{2,-n}\right)\left(1-\tilde{\psi}^{\dagger}_{1,-n}\psi_{2,-n}\right)\left(1+\psi^{\dagger}_{1,-n}\tilde{\psi}_{2,-n}\right)\times \\
& \times\left(1-\psi_{1,-n}\tilde{\psi}^{\dagger}_{2,-n}\right)|-+-+\rangle\,.
\end{aligned}
\tag{4.9}
$$

The boson boundary condition for $|D,N\rangle$ is given by

$$
X_L(0,\sigma)|D,N\rangle = -X_R(0,\sigma)|D,N\rangle \qquad Y_L(0,\sigma)|D,N\rangle = Y_R(0,\sigma)|D,N\rangle\,, \tag{4.10}
$$

which can be transcribed into the fermion boundary condition as

$$
\begin{aligned}
\psi_{1L}(0,\sigma)|D,N\rangle &= \psi^{\dagger}_{2R}(0,\sigma)|D,N\rangle, \qquad \psi_{2L}(0,\sigma)|D,N\rangle = \psi^{\dagger}_{1R}(0,\sigma)|D,N\rangle\,, \\
\psi^{\dagger}_{1L}(0,\sigma)|D,N\rangle &= -\psi_{2R}(0,\sigma)|D,N\rangle, \quad \psi^{\dagger}_{2L}(0,\sigma)|D,N\rangle = -\psi_{1R}(0,\sigma)|D,N\rangle\,.
\end{aligned}
\tag{4.11}
$$

The boundary state $|D,N\rangle$ is

$$
\begin{aligned}
|D,N\rangle_{NS} \,=\, \prod_{r=1/2}^{\infty} & \left(1-\tilde{\psi}^{\dagger}_{1,-r}\psi^{\dagger}_{2,-r}\right)\left(1+\psi^{\dagger}_{1,-r}\tilde{\psi}^{\dagger}_{2,-r}\right)\left(1+\tilde{\psi}_{1,-r}\psi_{2,-r}\right)\times \\
& \times\left(1-\psi_{1,-r}\tilde{\psi}_{2,-r}\right)|0\rangle\,,
\end{aligned}
\tag{4.12a}
$$

$$
\begin{aligned}
|D,N\rangle_{R} \,=\, \prod_{n=0}^{\infty} & \left(1-\tilde{\psi}^{\dagger}_{1,-n}\psi^{\dagger}_{2,-n}\right)\left(1+\psi^{\dagger}_{1,-n}\tilde{\psi}^{\dagger}_{2,-n}\right)\left(1+\tilde{\psi}_{1,-n}\psi_{2,-n}\right)\times \\
& \times\left(1-\psi_{1,-n}\tilde{\psi}_{2,-n}\right)|----\rangle\,.
\end{aligned}
\tag{4.12b}
$$

Finally we construct the boundary state $|D,D\rangle$. It obeys the boundary conditions for boson variables as follows

$$
X_L(0,\sigma)|D,D\rangle = -X_R(0,\sigma)|D,D\rangle, \quad Y_L(0,\sigma)|D,D\rangle = -Y_R(0,\sigma)|D,D\rangle\,. \tag{4.13}
$$

The corresponding boundary condition in the fermion theory is

$$
\begin{aligned}
\psi_{1L}(0,\sigma)|D,D\rangle &= i\psi_{1R}(0,\sigma)|D,D\rangle\,, \qquad \psi_{2L}(0,\sigma)|D,D\rangle = -i\psi_{2R}(0,\sigma)|D,D\rangle\,, \\
\psi^{\dagger}_{1L}(0,\sigma)|D,D\rangle &= i\psi^{\dagger}_{1R}(0,\sigma)|D,D\rangle\,, \qquad \psi^{\dagger}_{2L}(0,\sigma)|D,D\rangle = -i\psi^{\dagger}_{2R}(0,\sigma)|D,D\rangle\,.
\end{aligned}
\tag{4.14}
$$

A solution of these equations is the boundary state

$$
\begin{aligned}
|D,D\rangle_{NS} \,=\, \prod_{r=\frac{1}{2}}^{\infty} & \left(1+i\psi^{\dagger}_{1,-r}\tilde{\psi}_{1,-r}\right)\left(1+i\psi_{1,-r}\tilde{\psi}^{\dagger}_{1,-r}\right)\left(1-i\psi^{\dagger}_{2,-r}\tilde{\psi}_{2,-r}\right)\times \\
& \times\left(1-i\psi_{2,-r}\tilde{\psi}^{\dagger}_{2,-r}\right)|0\rangle\,,
\end{aligned}
\tag{4.15a}
$$

$$
\begin{aligned}
|D,D\rangle_R = & -i\prod_{n=0}^{\infty}\left(1+i\psi^\dagger_{1,-n}\tilde{\psi}_{1,-n}\right)\left(1+i\psi_{1,-n}\tilde{\psi}^\dagger_{1,-n}\right)\left(1-i\psi^\dagger_{2,-n}\tilde{\psi}_{2,-n}\right)\times \\
& \times\left(1-i\psi_{2,-n}\tilde{\psi}^\dagger_{2,-n}\right)|-+-+\rangle\,. \qquad (4.15b)
\end{aligned}
$$

The boundary states that we have constructed are related to each other by global $\mathrm{SU}(2)_L\times \mathrm{SU}(2)'_L$ rotations. The easiest way to see this is to note that the boundary conditions which they obey are related in this way. It is easy to check that

$$
|D,D\rangle = e^{-\pi i(J^1_0+J'^1_0)}|N,N\rangle = e^{-\pi i J^1_0}|N,D\rangle = e^{-\pi i J'^1_0}|D,N\rangle\,. \qquad (4.16)
$$

Note that, unlike the transformation which obtained the rolling tachyon boundary state in eq. (3.15), which was not unitary, the transformations in eq. (4.16) are unitary rotations.

4.2 Open string partition functions

In this subsection, we will consider the open string partition function for the boundary conditions described by the boundary state $|N,D\rangle$. We will show that the partition functions computed using the boson and the fermion representation of the boundary states are indeed identical.

The boundary state in boson variables is

$$
|N,D\rangle = \sum_{\pi_{1L},\pi_{1R}}\prod_{n=1}^{\infty}\exp\left(-\frac{1}{n}\left(\beta_{1,-n}\tilde{\beta}_{2,-n}+\beta_{2,-n}\tilde{\beta}_{1,-n}\right)\right)|\pi_{1L},\pi_{1R},-\pi_{1R},\,-\pi_{1L}\rangle\,. \qquad (4.17)
$$

Where $|\pi_{1L},\pi_{1R},\pi_{2L},\,\pi_{2R}\rangle = |\pi_{1L},\pi_{1R} > \otimes|\pi_{2L},\,\pi_{2R}\rangle$ is the direct product of vacuum states for ϕ_1 and ϕ_2 and we have taken the boundary conditions into account in constraining $\pi_{2L}=-\pi_{1R}$ and $\pi_{2R}=-\pi_{1L}$. The momenta π_{1L} and π_{1R} in eq. (4.17) can have either integer or half-odd-integer values and should be summed over those values which are appropriate to the compactified X and Y bosons. The constraints were given in eqs. (3.23a)–(3.23d). For the states in $|N.D\rangle$, where $\pi_{2L}=\pi_{1R}$ and $\pi_{2R}=\pi_{1L}$, these reduce to

$$
\pi_{1L}-\pi_{1R}=w_X\,,\qquad m_X=0\,,\qquad w_Y=0\,,\qquad \pi_{1L}+\pi_{1R}=m_Y\,. \qquad (4.18)
$$

Here, w_X and m_Y are integers and there are two possibilities: π_{1L} and π_{1R} are both integers (then w_X and m_Y are either both even or both odd), or π_{1L} and π_{1R} are both half-odd-integers (and one of m_X or w_Y is even and the other is odd).

The open string partition function for the $|N,D\rangle$ state is

$$
Z_{(ND|ND)}[q] = \langle N,D|\; q^{L_{10}+\tilde{L}_{10}+L_{20}+\tilde{L}_{20}}\;|N,D\rangle\,, \qquad (4.19)
$$

where

$$
L_{a0}+\tilde{L}_{a0} = \frac{1}{2}\left(\pi^2_{aL}+\pi^2_{aR}\right)+\sum_{n=1}^{\infty}\left(\beta_{a,-n}\beta_{a,n}+\tilde{\beta}_{a,-n}\tilde{\beta}_{a,n}\right)-\frac{1}{12}\,, \qquad (4.20)
$$

labeled by $a = 1, 2$ belonging to ϕ_1 or ϕ_2. The partition function is computed as

$$
\begin{aligned}
Z_{(ND|ND)}[q] &= \sum_{\pi_{1L},\pi_{1R}} q^{\left[\pi_{1L}^2+\pi_{1R}^2-\frac{1}{6}\right]} \left(\prod_{n=1} \frac{1}{1-q^{2n}}\right)^2 \\
&= \sum_{m,n\in\mathbb{Z}} \left(q^{m^2+n^2-\frac{1}{6}} + q^{(m+\frac{1}{2})^2+(n+\frac{1}{2})^2-\frac{1}{6}}\right) \left(\prod_{n=1}^{\infty} \frac{1}{1-q^{2n}}\right)^2 , \qquad (4.21)
\end{aligned}
$$

where we have explicitly indicated summations over the two possible sets of values of (π_{1L}, π_{1R}), where both are integers or both are half-odd-integers.

To confirm that the above factorizes into partition functions for a Neumann state and Dirichlet state, we would re-arrange the summations as

$$
\begin{aligned}
Z_{(ND|ND)}[q] &= \sum_{m,n\in\mathbb{Z}} q^{\frac{1}{2}(m-n)^2-\frac{1}{12}} \left(q^{\frac{1}{2}(m+n)^2-\frac{1}{12}} + q^{\frac{1}{2}(m+n+1)^2-\frac{1}{12}}\right) \left(\prod_{n=1}^{\infty} \frac{1}{1-q^{2n}}\right)^2 \\
&= \left(\sum_{m\in\mathbb{Z}} q^{\frac{1}{2}m^2-\frac{1}{12}} \prod_{n=1}^{\infty} \frac{1}{1-q^{2n}}\right)^2 = Z_{(N|N)}[q] Z_{(D|D)}[q] \qquad (4.22)
\end{aligned}
$$

where, in the last step, we observe that, if we keep $m - n$ fixed as we vary m and n, the values of $m + n$ and $m + n + 1$ together sweep over all of the integers. Because the X and Y bosons are compactified at the self-dual radius, the Neumann and Dirichlet partition functions are identical, thus the square in eq. (4.22).

We will now compute the open string partition function $Z_{(ND|ND)}[q]$ in the fermion theory. We must consider states both in the NS-NS and R-R sectors and project onto states where the total fermion number is even. In both R-R and NS-NS sectors, the total fermion number of the boundary states is already even and the latter projection is not needed. Using the expression for the boundary state derived in the previous subsection, we find:

- The partition function $Z_{(ND|ND)}[q]$ in the NS-NS sector:

$$
\begin{aligned}
Z^{NS}_{(ND|ND)}[q] &= q^{-\frac{1}{6}} \prod_{n=\frac{1}{2}}^{\infty} \left(1+q^{2n}\right)^4 \\
&= \sum_{m,n} q^{\left[(m^2+n^2)-\frac{1}{6}\right]} \left(\prod_{n=1}^{\infty} \frac{1}{1-q^{2n}}\right)^2 . \qquad (4.23)
\end{aligned}
$$

- The partition function $Z_{(ND|ND)}[q]$ in the R-R sector:

$$
\begin{aligned}
Z^{R}_{(ND|ND)}[q] &= 2^2 q^{\frac{1}{3}} \prod_{n=1}^{\infty} \left(1+q^{2n}\right)^4 \\
&= \sum_{m,n} q^{\left[\left((m+\frac{1}{2})^2+(n+\frac{1}{2})^2\right)-\frac{1}{6}\right]} \left(\prod_{n=1}^{\infty} \frac{1}{1-q^{2n}}\right)^2 . \qquad (4.24)
\end{aligned}
$$

JHEP05(2005)072

The total partition function is given as a sum of contributions from both sectors, $Z_{(ND|ND)}[q] = Z^{NS}_{(ND|ND)}[q] + Z^{R}_{(ND|ND)}[q]$. Upon combining the expressions in (4.23) and (4.24) we find precise agreement with the partition function in the boson theory eq. (4.21).

4.3 Exact boundary state for the rolling tachyon

Recall that the boundary state for the rolling tachyon is given in eq. (3.15), which we copy here:

$$|B,D\rangle = \exp\left(g\pi i J_0^+\right)|N,D\rangle\,, \qquad J_0^+ = \int \frac{d\sigma}{2\pi}\psi^\dagger_{1L}\psi_{2L} = \sum_n \psi^\dagger_{1,n}\psi_{2,-n} \tag{4.25}$$

with $n \in \mathbb{Z}+\frac{1}{2}$ in the NS-NS sector, $n \in \mathbb{Z}$ in the R-R sector. Combining (3.15) with eqs. (4.8) and (4.9), we find the explicit form of the boundary state is

$$\begin{aligned} |B,D\rangle_{NS} &= \prod_{r=\frac{1}{2}}^{\infty}\left(1+\psi^\dagger_{1,-r}\tilde\psi_{2,-r}\right)\left(1+\tilde\psi^\dagger_{2,-r}\psi_{1,-r} - i\pi g\tilde\psi^\dagger_{2,-r}\psi_{2,-r}\right)\cdot \\ &\qquad\cdot\left(1-\psi^\dagger_{2,-r}\tilde\psi_{1,-r} - i\pi g\psi^\dagger_{1,-r}\tilde\psi_{1,-r}\right)\left(1-\tilde\psi^\dagger_{1,-r}\psi_{2,-r}\right)|0\rangle \end{aligned} \tag{4.26a}$$

$$\begin{aligned} |B,D\rangle_{R} &= -i\prod_{n=0}^{\infty}\left(1+\psi^\dagger_{1,-n}\tilde\psi_{2,-n}\right)\left(1+\tilde\psi^\dagger_{2,-n}\psi_{1,-n} - i\pi g\tilde\psi^\dagger_{2,-n}\psi_{2,-n}\right)\cdot \\ &\qquad\cdot\left(1-\psi^\dagger_{2,-n}\tilde\psi_{1,-n} - i\pi g\psi^\dagger_{1,-n}\tilde\psi_{1,-n}\right)\left(1-\tilde\psi^\dagger_{1,-n}\psi_{2,-n}\right)|-+-+\rangle\,. \end{aligned} \tag{4.26b}$$

The formulae presented in eqs. (4.26a) and (4.26b) are our main result, the fermion representation of the boundary state of the rolling tachyon. Note that, in these states, the total fermion number is even, so they both correspond directly to boson states, with no further need of projection.

5. Open string partition function for the rolling tachyon

Now that we have an explicit expression for the boundary state in eqs. (4.26a) and (4.26b) we can form the open string partition function[2]

$$Z_{(BD|BD)}[q] = \langle B,D|q^{L_0+\tilde L_0}|B,D\rangle\,. \tag{5.1}$$

In the NS-NS sector this is given by

$$\begin{aligned} Z^{NS}_{(BD|BD)}[q] &= q^{-\frac{1}{6}}\prod_{r=\frac{1}{2}}^{\infty}\left[(1+\zeta q^{2r})(1+\zeta^{-1}q^{2r})\right]^2 \\ &= \left[q^{-\frac{1}{12}}\sum_{n\in\mathbb{Z}}\zeta^n q^{n^2}\prod_{n=1}^{\infty}\frac{1}{1-q^{2n}}\right]^2 \end{aligned} \tag{5.2}$$

[2]For this computation we use the identity $\left(1+\psi_1^\dagger\tilde\psi_2\right) = e^{\psi_1^\dagger\tilde\psi_2}$ to express the non-zero mode parts of the boundary states using exponential operators operating on the vacuum. Then we use the identity $\langle 0|e^{\psi^\dagger M\tilde\psi}e^{\tilde\psi^\dagger N\psi}|0\rangle = \det(1+MN)$.

$$= \left[\sum_{m,n \in \mathbb{Z}} q^{\frac{1}{2}(m-n)^2} \zeta^{(m+n)} q^{\frac{1}{2}(m+n)^2} \right] \left[q^{-\frac{1}{12}} \prod_{n=1}^{\infty} \frac{1}{1-q^{2n}} \right]^2 ,$$

where

$$\zeta = 1 + \frac{1}{2}\pi^2 g^2 + g\pi \sqrt{1 + \frac{1}{4}\pi^2 g^2}$$

and in the R-R sector,

$$\begin{aligned} Z^R_{(BD|BD)}[q] &= \left(\zeta^{\frac{1}{2}} + \zeta^{-\frac{1}{2}}\right)^2 q^{\frac{1}{3}} \prod_{m=1}^{\infty} \left[(1+\zeta q^{2m})(1+\zeta^{-1} q^{2m})\right]^2 \\ &= \left[q^{-\frac{1}{12}} \sum_{n \in \mathbb{Z}} \zeta^{n+\frac{1}{2}} q^{(n+\frac{1}{2})^2} \prod_{n=1}^{\infty} \frac{1}{1-q^{2n}} \right]^2 \\ &= \left[\sum_{m,n \in \mathbb{Z}} q^{\frac{1}{2}(m-n)^2} \zeta^{(m+n+1)} q^{\frac{1}{2}(m+n+1)^2} \right] \left[q^{-\frac{1}{12}} \prod_{n=1}^{\infty} \frac{1}{1-q^{2n}} \right]^2 . \end{aligned} \tag{5.3}$$

It is easy to see that combining (5.2) and (5.3) obtains a summation over all integers $m-n$ where, holding $m-n$ fixed, $m+n$ and $m+n+1$ together also take on all integer values. Thus, we get

$$Z^{NS}_{(BD|BD)}[q] + Z^R_{(BD|BD)}[q] = Z_{DD}[q] \cdot \left[\sum_{n \in \mathbb{Z}} \zeta^n q^{\frac{1}{2}n^2} \right] \left[q^{-\frac{1}{12}} \prod_{n=1}^{\infty} \frac{1}{1-q^{2n}} \right] , \tag{5.4}$$

where $Z_{DD}[q]$ is the Dirichlet partition function of the Y-boson and our result for the rolling tachyon partition function is

$$Z_{BB}[q] = q^{-\frac{1}{12}} \sum_{n \in \mathbb{Z}} \zeta^n q^{\frac{1}{2}n^2} \prod_{n=1}^{\infty} \frac{1}{1-q^{2n}} . \tag{5.5}$$

6. Disc amplitude and time evolution of the rolling tachyon

We have found a fermion representation of the states of compact bosons. However, the original problem of the rolling tachyon that we set out to solve was in Minkowski space. The rotation to euclidean space should not produce a compact boson. Instead, our computations should be regarded as being incomplete, we should discard all wrapped states and we have only found a subset of the continuous spectrum of momentum states of the non-compact boson. We might hope, nevertheless, that the discrete sampling of momentum states that we have found are enough to determine the spectrum.

For example, recall that the momentum and wrappping integers for the X and Y-bosons are given by (m_X, w_X, m_Y, w_Y) which in turn are the combinations of the momenta $(\pi_{1L}, \pi_{1R}, \pi_{2L}, \pi_{2R})$ .given in eqs. (3.23a)–(3.23d). To set the two wrapping numbers to zero, we set $\pi_{1L} = \pi_{1R} \equiv \pi_1$ and $\pi_{2L} = \pi_{2R} \equiv \pi_2$. Then $(m_X, w_X, m_Y, w_Y) = (\pi_1 + \pi_2, 0, \pi_1 - \pi_2, 0)$.

Consider the matrix element of this state with the $|N,D\rangle$ boundary state as given in the bosonic theory (4.17).

$$\langle \pi_1, \pi_1, \pi_2, \pi_2 | N, D\rangle = \delta_{\pi_1+\pi_2} \,. \tag{6.1}$$

If we define the position space state,

$$|X, Y\rangle = \frac{1}{2\pi} \sum_{\pi_1,\pi_2} e^{-iX(\pi_1+\pi_2)-iY(\pi_1-\pi_2)} |\pi_1, \pi_1, \pi_2, \pi_2\rangle \tag{6.2}$$

we see that

$$\langle X, Y | N, D\rangle = \sum_{\pi_1,\pi_2} e^{iX(\pi_1+\pi_2)+iY(\pi_1-\pi_2)} \delta_{\pi_1,-\pi_2} = \sum_n \delta(Y + 2\pi n) \tag{6.3}$$

which is what we would expect, given the Dirichlet boundary condition for Y. Note that this formula retains the information that we treated Y as a compact boson.

In the following, we will learn how to obtain the same result in the fermion representation. Then, we will compute $\langle X, Y | B, D\rangle$ using the fermion theory and compare it with the known result of the current algebra computation (3.34).

6.1 Momentum states

To begin, we must find the fermion representation of the boson states which are oscillator vacua and eigenstates of the momentum operators, $|\pi_{1L}, \pi_{1R}, \pi_{2L}, \pi_{2R}\rangle$. In the fermion representation in eqs. (3.10a) and (3.10b),

$$\pi_{1L} = \oint \frac{d\sigma}{2\pi} : \psi^\dagger_{1L}(0,\sigma)\psi_{1L}(0,\sigma) : , \qquad \pi_{1R} = -\oint \frac{d\sigma}{2\pi} : \psi^\dagger_{1R}(0,\sigma)\psi_{1R}(0,\sigma) : \tag{6.4a}$$

$$\pi_{2L} = -\oint \frac{d\sigma}{2\pi} : \psi^\dagger_{2L}(0,\sigma)\psi_{2L}(0,\sigma) : , \qquad \pi_{2R} = \oint \frac{d\sigma}{2\pi} : \psi^\dagger_{2R}(0,\sigma)\psi_{2R}(0,\sigma) : . \tag{6.4b}$$

The state $|\pi_{1L}, \pi_{1R}, \pi_{2L}, \pi_{2R}\rangle$ is an eigenstate of fermion number operators and, since it is a vacuum for the boson oscillators, it must be annihilated by the positively moded boson oscillators. These correspond to the positively moded components of the fermion charge densities, which can also be found from eqs. (3.10a) and (3.10b),

$$\beta_{1,n} = \sum_r \psi^\dagger_{1,n-r}\psi_{1,r} \ , \qquad \tilde\beta_{1,n} = -\sum_r \tilde\psi^\dagger_{1,n-r}\tilde\psi_{1,r} \,, \tag{6.5a}$$

$$\beta_{2,n} = -\sum_r \psi^\dagger_{2,n-r}\psi_{2,r} \,, \qquad \tilde\beta_{2,n} = \sum_r \tilde\psi^\dagger_{2,n-r}\tilde\psi_{2,r} \,, \qquad n > 0 \,. \tag{6.5b}$$

For example, the unique eigenstate of $L_0 + \tilde L_0$ with the lowest eigenvalue is the state $|0,0,0,0\rangle$ of the bosons which is annihilated by all positively moded boson oscillators. This state coincides with the vacuum of the fermions in the NS-NS sector, $|0\rangle$ which is annihilated by all positively moded fermion oscillators.

States with nonzero integer momenta are obtained by creating fermions to fill up all fermion states up to a specific fermi level, which determines the fermion number of the state and therefore its momentum. For example, the state where π_{1L} is a positive integer is given by (up to a phase)

$$|\pi_{1L},0,0,0\rangle \ \sim\ \psi^\dagger_{1,-\pi_{1L}+\frac{1}{2}}\cdots\psi^\dagger_{1,-\frac{3}{2}}\psi^\dagger_{1,-\frac{1}{2}}\ |0\rangle\,.$$

If π_{1L} is a negative, rather than positive integer, the state is

$$|\pi_{1L},0,0,0\rangle \ \sim\ \psi_{1,\pi_{1L}+\frac{1}{2}}\cdots\psi_{1,-\frac{3}{2}}\psi_{1,-\frac{1}{2}}\ |0\rangle\,.$$

These states satisfy the conditions

$$\sum_{r=\frac{1}{2}}^{\infty}\left(\psi^\dagger_{1,-r}\psi_{1,r}-\psi_{1,-r}\psi^\dagger_{1,r}\right)|\pi_{1L},0,0,0\rangle \ =\ \pi_{1L}|\pi_{1L},0,0,0\rangle \tag{6.6a}$$

$$\sum_{r=-\infty}^{\infty}\psi^\dagger_{1,n-r}\psi_{1,r}|\pi_{1L},0,0,0\rangle \ =\ 0 \qquad n>0 \tag{6.6b}$$

and thus coincide with the requisite states in the boson theory, up to a phase which is not determined by these equations.

In the boson representation, in the phase convention that we have chosen for the boundary state in eqn. (4.17), the inner products of momentum eigenstates and boundary states are all either one or zero. We have found a phase convention which produces this result in the fermion representation for the momentum states of interest to us:

$$\begin{cases}\langle 0|[\psi_{1,\frac{1}{2}}\cdots\psi_{1,\pi_1-\frac{1}{2}}][\tilde\psi^\dagger_{1,\pi_1-\frac{1}{2}}\cdots\tilde\psi^\dagger_{1,\frac{1}{2}}][\psi^\dagger_{2,\frac{1}{2}}\cdots\psi^\dagger_{2,\pi_2-\frac{1}{2}}][\tilde\psi_{2,\pi_2-\frac{1}{2}}\cdots\tilde\psi_{2,\frac{1}{2}}]\ i^{\pi_1-\pi_2} & \pi_1>0,\pi_2>0,\\ \langle 0|[\psi_{1,\frac{1}{2}}\cdots\psi_{1,\pi_1-\frac{1}{2}}][\tilde\psi^\dagger_{1,\pi_1-\frac{1}{2}}\cdots\tilde\psi^\dagger_{1,\frac{1}{2}}][\psi_{2,\frac{1}{2}}\cdots\psi_{2,-\pi_2-\frac{1}{2}}][\tilde\psi^\dagger_{2,-\pi_2-\frac{1}{2}}\cdots\tilde\psi^\dagger_{2,\frac{1}{2}}]\ i^{\pi_1+\pi_2} & \pi_1>0,\pi_2<0\\ \langle 0|[\psi^\dagger_{1,\frac{1}{2}}\cdots\psi^\dagger_{1,-\pi_1-\frac{1}{2}}][\tilde\psi_{1,-\pi_1-\frac{1}{2}}\cdots\tilde\psi_{1,\frac{1}{2}}][\psi_{2,\frac{1}{2}}\cdots\psi_{2,-\pi_2-\frac{1}{2}}][\tilde\psi^\dagger_{2,-\pi_2-\frac{1}{2}}\cdots\tilde\psi^\dagger_{2,\frac{1}{2}}]\ i^{-\pi_1+\pi_2} & \pi_1<0,\pi_2<0\\ \langle 0|[\psi^\dagger_{1,\frac{1}{2}}\cdots\psi^\dagger_{1,-\pi_1-\frac{1}{2}}][\tilde\psi_{1,-\pi_1-\frac{1}{2}}\cdots\tilde\psi_{1,\frac{1}{2}}][\psi^\dagger_{2,\frac{1}{2}}\cdots\psi^\dagger_{2,\pi_2-\frac{1}{2}}][\tilde\psi_{2,\pi_2-\frac{1}{2}}\cdots\tilde\psi_{2,\frac{1}{2}}]\ i^{-\pi_1-\pi_2} & \pi_1<0,\pi_2>0\end{cases} \tag{6.7}$$

The additional phases which occur with each state are necessary to make the matrix elements of each state with the simply fermionic boundary states $|N,N\rangle,|D,N\rangle,|N,D\rangle$, $|D,D\rangle$ match those of the bosonic theory. In the boson theory, the overlap of any momentum eigenstate with the simple boundary states is either zero or one, depending on the momenta.

The half-odd-integer momenta are found in the R-R sector. There, the choice of degenerate fermion vacuum is determined by the requirement that the state be a boson oscillator vacuum. For example, the state with $\pi_{1L}>0$ is

$$\psi^\dagger_{1,-\pi_{1L}+\frac{1}{2}}\cdots\psi^\dagger_{1,-2}\psi^\dagger_{1,-1}|+\rangle$$

whereas, with $\pi_{1L}<0$,

$$\psi_{1,\pi_{1L}+\frac{1}{2}}\cdots\psi_{1,-2}\psi_{1,-1}|-\rangle\,.$$

Again, our phase conventions can be summarized in bra-vector form below

$$\begin{cases}\langle +--+|[\psi_{1,1}\ldots\psi_{1,\pi_1-\frac{1}{2}}][\tilde\psi^\dagger_{1,\pi_1-\frac{1}{2}}\ldots\tilde\psi^\dagger_{1,1}]\ [\psi^\dagger_{2,1}\ldots\psi^\dagger_{2,\pi_2-\frac{1}{2}}][\tilde\psi_{2,\pi_2-\frac{1}{2}}\ldots\tilde\psi_{2,1}]\ i^{\pi_1-\pi_2} & \pi_1>0,\pi_2>0\\ \langle +-+-|[\psi_{1,1}\ldots\psi_{1,\pi_1-\frac{1}{2}}][\tilde\psi^\dagger_{1,\pi_1-\frac{1}{2}}\ldots\tilde\psi^\dagger_{1,1}][\psi_{2,1}\ldots\psi_{2,-\pi_2-\frac{1}{2}}][\tilde\psi^\dagger_{2,-\pi_2-\frac{1}{2}}\ldots\tilde\psi^\dagger_{2,1}]\ i^{\pi_1+\pi_2+1} & \pi_1>0,\pi_1<0\\ \langle -+-+|[\psi^\dagger_{1,1}\ldots\psi^\dagger_{1,-\pi_1-\frac{1}{2}}][\tilde\psi_{1,-\pi_1-\frac{1}{2}}\ldots\tilde\psi_{1,1}][\psi^\dagger_{2,1}\ldots\psi^\dagger_{2,\pi_2-\frac{1}{2}}][\tilde\psi_{2,\pi_2-\frac{1}{2}}\ldots\tilde\psi_{2,1}]\ i^{-\pi_1-\pi_2+1} & \pi_1<0,\pi_1>0\\ \langle -++-\ |[\psi^\dagger_{1,1}\ldots\psi^\dagger_{1,-\pi_1-\frac{1}{2}}][\tilde\psi_{1,-\pi_1-\frac{1}{2}}\ldots\tilde\psi_{1,1}][\psi_{2,1}\ldots\psi_{2,-\pi_2-\frac{1}{2}}][\tilde\psi^\dagger_{2,-\pi_2-\frac{1}{2}}\ldots\tilde\psi^\dagger_{2,1}]\ i^{-\pi_1+\pi_2+2} & \pi_1<0,\pi_2<0\,.\end{cases} \tag{6.8}$$

Here, the ordering of the symbols in the vacuum bra-vector, for example $\langle - - - - |$ follows the same convention as for ket-vectors, the first is the state of $\psi_{1,0}$, the second $\tilde{\psi}_{1,0}$, etc., so that $\langle - - - - |\psi_{1,0} = \langle + - - - |$. Again, the phases of these states have been adjusted so that they have the correct overlaps with the simple boundary states.

It is easy to check that, in both the NS and R sectors, the matrix elements of our momentum eigenstates with boundary states are

$$\begin{aligned}\langle \pi_1, \pi_1, \pi_2, \pi_2 | N, N\rangle &= \delta_{\pi_1}\delta_{\pi_2} \\ \langle \pi_1, \pi_1, \pi_2, \pi_2 | N, D\rangle &= \delta_{\pi_1+\pi_2} \\ \langle \pi_1, \pi_1, \pi_2, \pi_2 | D, N\rangle &= \delta_{\pi_1-\pi_2} \\ \langle \pi_1, \pi_1, \pi_2, \pi_2 | D, D\rangle &= 1\,. \end{aligned} \tag{6.9}$$

This agrees with what we would obtain in the boson theory.

6.2 Rolling tachyon

Now, we are ready to compute the matrix elements of the momentum eigenstates with the rolling tachyon state. There are three cases of relative signs of the momenta. First, when $\pi_1, \pi_2 \geq 0$,

$$\langle \pi_1, \pi_1, \pi_2, \pi_2 | B, D\rangle = (-\pi g)^{\pi_1}(-\pi g)^{\pi_2}\,. \tag{6.10}$$

Second, when $\pi_1 \geq 0$ and $\pi_2 \leq 0$ and $\pi_2 + \pi_1 \geq 0$,

$$\langle \pi_1, \pi_1, \pi_2, \pi_2 | B, D\rangle = (-\pi g)^{\pi_1+\pi_2}\delta_{\pi_1+\pi_2}\,. \tag{6.11}$$

Third, when $\pi_1 \leq 0$ and $\pi_2 \geq 0$ and $\pi_2 + \pi_1 \geq 0$

$$\langle \pi_1, \pi_1, \pi_2, \pi_2 | B, D\rangle = (-\pi g)^{\pi_1+\pi_2}\delta_{\pi_{1R}+\pi_{2L}}\,. \tag{6.12}$$

Finally,

$$\langle \pi_1, \pi_1, \pi_2, \pi_2 | B, D\rangle = 0 \tag{6.13}$$

otherwise.

The above formulae apply to both the NS and R sectors where, in the former the momenta are integers and in the latter they are half-odd-integers. It is then straightforward to find the disc amplitude of the rolling tachyon. We take the sums

$$\begin{aligned}\langle X, Y | B, D\rangle &= \sum_{\pi_i,\pi_2} e^{iX(\pi_1+\pi_2)+iY(\pi_1-\pi_2)}\langle \pi_1, \pi_1, \pi_2\pi_2 | B, D\rangle \\ &= \sum_{\pi_1>0,\pi_2>0}\left(-\pi g e^{iX}\right)^{\pi_1+\pi_2} e^{iY(\pi_1-\pi_2)} + \sum_{\pi_1=0}^{-\infty}\sum_{\pi_2=-\pi_1}^{\infty}\left(-\pi g e^{iX}\right)^{\pi_1+\pi_2} e^{iY(\pi_1-\pi_2)} + \\ &\quad + \sum_{\pi_2=0}^{-\infty}\sum_{\pi_1=-\pi_2}^{\infty}\left(-\pi g e^{iX}\right)^{\pi_1+\pi_2} e^{iY(\pi_1-\pi_2)} \\ &= \sum_{\pi_1+\pi_1=0}^{\infty}\sum_{\pi_1-\pi_2=-\infty}^{\infty}\left(-\pi g e^{iX}\right)^{\pi_1+\pi_2} e^{iY(\pi_1-\pi_2)} = \frac{1}{1+\pi g e^{iX}}\sum_n \delta(Y+2\pi n)\,. \end{aligned} \tag{6.14}$$

Once we remove the amplitude for the Y-boson, and analytically continue back to euclidean space, $X \to -it$, we find that the disc amplitude is

$$\langle t|B\rangle = \frac{1}{1 + \pi g e^t}$$

which agrees with the result of current algebra, quoted in (3.34).

7. Conclusion

Time evolution of the unstable D-brane, often termed as the rolling tachyon in string theory, is the key to understand the dynamics of the string theory. Without understanding of this time dependent process the string theory should be regarded as incomplete. The most useful framework to study the time evolution of the unstable D-brane may be the boundary state formulation, which describes most efficiently the interaction between the open string and a D-brane. The tachyon mode of the open string coordinate along the time like direction of the target space induces a nontrivial interaction on the unstable D-brane. Upon taking the Wick rotation on the string coordinate we find that the system is be described by a boundary conformal theory with a marginal periodic boundary interaction. The theory for the Wick rotated system turns out to be the boundary conformal theory which has a wide range of applications in condensed matter physics [4, 5] such as the Kondo model and junctions in quantum wires. Thanks to the previous study of the system [4, 5] the exact boundary state is known as a superposition of the SU(2) Ishibashi states. However, the complete description of the time evolution and the final fate of the unstable D-brane is still out of reach of the current study since the transition functions between the Ishibashi states and the perturbative string states are not known in general except for a few states at low mass levels. Thus, there is an urgent need for a more suitable perturbative basis with which the exact boundary state may be written explicitly. This is the main purpose of the present work.

With the help of the extra boson field the entire system of the unstable D-brane can be fermionized. The advantage of the fermionization is that the boundary interaction can be written as a bilinear operator, being a SU(2) current operator, in the fermionized theory. It readily implies that the boundary condition to be satisfied can be linear in terms of the fermion fields. Taking advantage of the fermionization we are able to construct the exact boundary state for the D-brane, which takes a form of squeezed state in the fermion theory. The fermionization or bosonization of the conformal system with the marginal boundary interaction is not new. Once the auxiliary boson field Y is introduced, the system becomes the bosonized Kondo model [46, 47] which has been extensively studied in connection with various phenomena in condensed matter physics. The boundary conformal field theory of the rolling tachyon in the fermionized form is in fact identical to the Kondo model in its original form. The fermionization of the same conformal model has been also considered by Polchinski and Thorlacius. But their study was limited to the open string theory.

In this Paper we develop a closed string version of the fermionized conformal theory, which is more suitable to discuss the rolling tachyon and construct the exact boundary state. Calculating the space-time dependent disk amplitude with a scalar vertex explicitly we prove that the fermion perturbative basis is the most suitable one to investigate the time evolution of the unstable D-brane at every level. In this work we mainly deal with the simplest case of the half-S-brane. Certainly, the time evolution of the rolling tachyon with more general tachyon profiles can be discussed in this fermion representation developed here. We may choose $|N, N\rangle$ as the boundary state to start with to study the S-brane within this framework. The present work can be extended along various directions: The immediate one would be the rolling tachyon with electric and magnetic fields [25]. It is also interesting to apply the fermionization to the supersymmetric rolling tachyon. The outstanding problem is the complete description of the time evolution of the rolling tachyon at all higher levels and its final fate.

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A. Cocycles for fermion operators

A.1 One dimensional system

We parametrize the cocycles for the fermions as follows

$$
\begin{aligned}
\psi_L &= e^{-\frac{\pi i}{2}(\alpha^L p_L + \beta^L p_R)} e^{-\sqrt{2} i X_L}\,, \\
\psi_R &= e^{\frac{\pi i}{2}(\alpha^R p_L + \beta^R p_R)} e^{\sqrt{2} i X_R}\,.
\end{aligned}
$$

Their conjugates are

$$
\begin{aligned}
\psi_L^\dagger &= e^{-\frac{\pi i}{2}\alpha^L} e^{\frac{\pi i}{2}(\alpha^L p_L + \beta^L p_R)} e^{\sqrt{2} i X_L}\,, \\
\psi_R^\dagger &= e^{-\frac{\pi i}{2}\beta^R} e^{-\frac{\pi i}{2}(\alpha^R p_L + \beta^R p_R)} e^{-\sqrt{2} i X_R}\,,
\end{aligned}
$$

where we make use of

$$
[p_L, X_L] = [p_R, X_R] = -i\,.
$$

Here the left and right-moving bosons are operators defined by the mode expansion:

$$
\begin{aligned}
X_L(\tau + i\sigma) &= \frac{x_L}{\sqrt{2}} - \frac{i}{\sqrt{2}} p_L(\tau + i\sigma) + \frac{i}{\sqrt{2}} \sum_{n\neq 0} \frac{1}{n} \alpha_n e^{-n(\tau + i\sigma)} \\
X_R(\tau - i\sigma) &= \frac{x_R}{\sqrt{2}} - \frac{i}{\sqrt{2}} p_R(\tau - i\sigma) + \frac{i}{\sqrt{2}} \sum_{n\neq 0} \frac{1}{n} \tilde{\alpha}_n e^{-n(\tau - i\sigma)}\,.
\end{aligned}
$$

JHEP05(2005)072

If the fermion field ψ_L anti-commutes with the fermion field ψ_R, all anti-commutation relations between the fermion operators are satisfied. It yields the following condition

$$e^{\frac{\pi i}{2}(\beta^L-\alpha^R)} = -1\,. \tag{A.1}$$

Thus, we need to impose

$$\beta^L - \alpha^R = 2(2n+1)\,, \qquad n \in \mathbb{Z}\,. \tag{A.2}$$

The Neumann state $|N\rangle$ is defined as

$$X_L(i\sigma)|N\rangle = X_R(-i\sigma)|N\rangle\,, \tag{A.3}$$

which reads in terms of normal mode operators as

$$x_L|N\rangle = x_R|N\rangle\,, \qquad p_L|N\rangle = -p_R|N\rangle, \quad \alpha_n|N\rangle = -\tilde{\alpha}_{-n}|N\rangle\,.$$

If we choose the cocycles appropriately we can simply realize the Neumann state in terms of fermion operators. It requires

$$\alpha^L - \alpha^R - \beta^L + \beta^R = 0\,. \tag{A.4}$$

Under this condition eq. (A.4) the Neumann state can be expressed in terms of fermion fields as follows

$$\psi_L|N\rangle = e^{\frac{\pi i}{2}\beta^L}\psi_R^\dagger|N\rangle, \quad \psi_L^\dagger|N\rangle = e^{\frac{\pi i}{2}(-\alpha^L+\beta^L-\beta^R)}\psi_R|N\rangle\,. \tag{A.5}$$

This condition is consistent only if

$$e^{-\frac{\pi i}{2}(\alpha^L-2\beta^L+\beta^R)} = -1\,. \tag{A.6}$$

We also require that the Dirichlet state $|D\rangle$ takes a simple form in the fermion theory. The Dirichlet condition is given in the bosonic theory as

$$X_L|D\rangle = -X_R|D\rangle\,, \tag{A.7}$$

which can be read in terms of normal modes as

$$x_L|D\rangle = -x_R|D\rangle\,, \qquad p_L|D\rangle = p_R|D\rangle, \quad \alpha_n|D\rangle = \tilde{\alpha}_{-n}|D\rangle\,.$$

Applying the fermion operators on the Dirichlet state, we see that the Dirichlet condition is expressed in terms of the fermion operators only (without a nontrivial cocycle) only if

$$\alpha^L + \alpha^R + \beta^L + \beta^R = 0\,. \tag{A.8}$$

If this condition eq. (A.8) holds, the Dirichlet boundary condition in the fermion theory is given by

$$\psi_L|D\rangle = e^{\frac{\pi i}{2}(\alpha^L+\alpha^R)}\psi_R|D\rangle, \quad \psi_L^\dagger|D\rangle = e^{-\frac{\pi i}{2}(\alpha^L+\beta^L)}\psi_R^\dagger|D\rangle\,. \tag{A.9}$$

JHEP05(2005)072

This condition is consistent only if

$$e^{\frac{\pi i}{2}(\alpha^R-\beta^L)} = -1\,. \tag{A.10}$$

Note that this condition coincides with the anti-commutation condition eq. (A.2).

The solutions to the eqs. (A.2), (A.4), (A.6), (A.8) exist but not unique

$$\alpha^L = n\,, \qquad \beta^L = 2m+1\,, \qquad \alpha^R = -2m-1\,, \qquad \beta^R = -n\,, \tag{A.11}$$

where $n, m \in Z$. The simplest solution may be with $n = m = 0$

$$\alpha^L = 0\,, \qquad \beta^L = 1\,, \qquad \alpha^R = -1\,, \qquad \beta^R = 0\,. \tag{A.12}$$

With this choice the fermion field operators are defined as

$$\begin{aligned}\psi_L &= e^{-\frac{\pi i}{2}p_R}e^{-\sqrt{2}iX_L}\,, \qquad \psi_R = e^{-\frac{\pi i}{2}p_L}e^{\sqrt{2}iX_R}\\ \psi_L^\dagger &= e^{\frac{\pi i}{2}p_R}e^{\sqrt{2}iX_L}\,, \qquad \psi_R^\dagger = e^{\frac{\pi i}{2}p_L}e^{-\sqrt{2}iX_R}\,.\end{aligned} \tag{A.13}$$

Accordingly the Neumann boundary condition and the Dirichlet boundary condition are written as

$$\begin{aligned}\psi_L|N\rangle &= i\psi_R^\dagger|N\rangle\,, \qquad \psi_L^\dagger|N\rangle = i\psi_R|N\rangle\,,\\ \psi_L|D\rangle &= -i\psi_R|D\rangle\,, \qquad \psi_L^\dagger|D\rangle = -i\psi_R^\dagger|D\rangle\,.\end{aligned} \tag{A.14}$$

A.2 Two dimensional system

We may parametrize the cocycles for the two dimensional system with two Dirac fermions in general as follows

$$\begin{aligned}\psi_{1L} &= e^{-\frac{\pi i}{2}(\alpha^L_{1j}p^j_L+\beta^L_{1j}p^j_R)}e^{-\sqrt{2}i\Phi^1_L}\,,\\ \psi_{2L} &= e^{\frac{\pi i}{2}(\alpha^L_{2j}p^j_L+\beta^L_{2j}p^j_R)}e^{\sqrt{2}i\Phi^2_L}\,,\\ \psi_{1R} &= e^{\frac{\pi i}{2}(\alpha^R_{1j}p^j_L+\beta^R_{1j}p^j_R)}e^{\sqrt{2}i\Phi^1_R}\,,\\ \psi_{2R} &= e^{-\frac{\pi i}{2}(\alpha^R_{2j}p^j_L+\beta^R_{2j}p^j_R)}e^{-\sqrt{2}i\Phi^2_R}\,,\end{aligned} \tag{A.15}$$

where $j = 1, 2$. Their conjugates are

$$\begin{aligned}\psi^\dagger_{1L} &= e^{-\frac{\pi i}{2}\alpha^L_{11}}e^{\frac{\pi i}{2}(\alpha^L_{1j}p^j_L+\beta^L_{1j}p^j_R)}e^{\sqrt{2}i\Phi^1_L}\,,\\ \psi^\dagger_{2L} &= e^{-\frac{\pi i}{2}\alpha^L_{22}}e^{-\frac{\pi i}{2}(\alpha^L_{2j}p^j_L+\beta^L_{2j}p^j_R)}e^{-\sqrt{2}i\Phi^2_L}\,,\\ \psi^\dagger_{1R} &= e^{-\frac{\pi i}{2}\beta^R_{11}}e^{-\frac{\pi i}{2}(\alpha^R_{1j}p^j_L+\beta^R_{1j}p^j_R)}e^{-\sqrt{2}i\Phi^1_R}\,,\\ \psi^\dagger_{2R} &= e^{-\frac{\pi i}{2}\beta^R_{22}}e^{\frac{\pi i}{2}(\alpha^R_{2j}p^j_L+\beta^R_{2j}p^j_R)}e^{\sqrt{2}i\Phi^2_R}\,,\end{aligned} \tag{A.16}$$

where we make use of

$$[p^i_L, \Phi^j_L] = [p^i_R, \Phi^j_R] = -i\delta^{ij}\,. \tag{A.17}$$

Requiring the anti-commutation relation between ψ_{iL} and ψ_{jL} we find

$$e^{\frac{\pi i}{2}(\alpha^L_{ij}-\alpha^L_{ji})} = -1, \qquad \text{for} \qquad i \neq j\,. \tag{A.18}$$

Likewise requiring the anti-commutation relation between ψ_{iR} and ψ_{jR}, we get

$$e^{\frac{\pi i}{2}(\beta^R_{ij}-\beta^R_{ji})} = -1\,, \qquad \text{for} \qquad i \neq j\,. \tag{A.19}$$

The anti-commutation relation between ψ_{iL} and ψ_{jR} yields the following condition

$$e^{\frac{\pi i}{2}(\beta^L_{ij}-\alpha^R_{ji})} = -1\,. \tag{A.20}$$

Then anti-commutation relations will be ensured between all fermion operators.

The simple boundary conditions should be realized in terms of the fermion operators (without the cocycles). We begin with the boundary state $|N,N\rangle$. The $|N,N\rangle$ boundary condition is given in terms of the bosonic operator as follows

$$\Phi^i_L|N,N\rangle = \Phi^i_R|N,N\rangle\,. \tag{A.21}$$

This condition can be realized simply in fermion theory if the following conditions are satisfied

$$\alpha^L_{ij} - \alpha^R_{ij} - \beta^L_{ij} + \beta^R_{ij} = 0\,. \tag{A.22}$$

Under this condition we may write define $|N,N\rangle$ in the fermion theory as

$$\psi^i_L|N,N\rangle = e^{\frac{\pi i}{2}\beta^L_{ii}}\psi^{i\dagger}_R|N,N\rangle\,, \qquad \psi^{i\dagger}_L|N,N\rangle = e^{-\frac{\pi i}{2}(\alpha^L_{ii}-\beta^L_{ii}+\beta^R_{ii})}\psi^i_R|N,N\rangle\,. \tag{A.23}$$

The consistency requires

$$e^{\frac{\pi i}{2}(-\alpha^L_{ii}+2\beta^L_{ii}-\beta^R_{ii})} = -1\,. \tag{A.24}$$

In the two dimensional system we have well defined fermion current operators J^1 and J'^1

$$\begin{aligned} e^{-\pi i J^1_0} J^3 e^{\pi i J^1_0} &= e^{-\pi i J^1_0} i\partial_z X e^{\pi i J^1_0} = -i\partial_z X\,, \\ e^{-\pi i J'^1_0} J'^3 e^{\pi i J'^1_0} &= e^{-\pi i J'^1_0} i\partial_z Y e^{\pi i J'^1_0} = -i\partial_z Y\,. \end{aligned} \tag{A.25}$$

Using these fermion current operators we can obtain the boundary states $|N,D\rangle$ $|D,N\rangle$ and $|D,D\rangle$ consistently if we once have the boundary state $|N,N\rangle$ in the fermion theory

$$|D,N\rangle = e^{-\pi i J^1_0}|N,N\rangle\,, \qquad |N,D\rangle = e^{-\pi i J'^1_0}|N,N\rangle\,, \qquad |D,D\rangle = e^{-\pi i(J^1_0+J'^1_0)}|N,N\rangle\,. \tag{A.26}$$

The tachyon boundary interaction term may be written in terms of fermion operators only if we choose $\alpha's$ and $\beta's$ appropriately. Since we may write the tachyon interaction term

$$: e^{i(X_L+X_R)} : |B,D\rangle =: e^{i(X_L+Y_L)} :: e^{i(X_R+Y_R)} : |B,D\rangle\,, \tag{A.27}$$

we find that it can be rewritten in terms of fermion field operators only if the following condition is satisfied

$$\alpha^L_{1j} + \alpha^R_{1j} = 0\,, \qquad \beta^L_{1j} + \beta^R_{1j} = 0\,. \tag{A.28}$$

Then the tachyon boundary interaction term can be written as

$$: e^{i(X_L+X_R)} : |B,D\rangle = e^{\frac{\pi i}{2}(\alpha^L_{11}+\alpha^R_{11})}\psi^\dagger_{1L}\psi_{1R}|B,D\rangle = \psi^\dagger_{1L}\psi_{1R}|B,D\rangle\,. \tag{A.29}$$

We note that the tachyon interaction term can be also written as

$$:e^{i(X_L+X_R)}:|B,D\rangle =: e^{i(X_L-Y_L)} :: e^{i(X_R-Y_R)}:|B,D\rangle\,. \tag{A.30}$$

This equation can be expressed in terms of the fermion field operators if the following condition is satisfied

$$\alpha^L_{2j}+\alpha^R_{2j}=0\,,\qquad \beta^L_{2j}+\beta^R_{2j}=0\,. \tag{A.31}$$

Under this condition eq. (A.31) the tachyon boundary interaction term can be written as

$$:e^{i(X_L+X_R)}:|B,D\rangle = e^{\frac{\pi i}{2}(\beta^L_{22}+\beta^R_{22})}\psi^\dagger_{2R}\psi_{2L}|B,D\rangle = \psi^\dagger_{2R}\psi_{2L}|B,D\rangle\,. \tag{A.32}$$

It is not difficult to show that there exist solutions to the conditions eqs. (A.18)–(A.20), (A.22), (A.24), (A.28), (A.31).

$$\begin{aligned}
\alpha^L_{11} &= \beta^L_{11} = -\alpha^R_{11} = -\beta^R_{11} = 2n+1\,,\\
\alpha^L_{12} &= \beta^L_{12} = -\alpha^R_{12} = -\beta^R_{12} = 2k+2l+2\,,\\
\alpha^L_{21} &= \beta^L_{21} = -\alpha^R_{21} = -\beta^R_{21} = 2k-2l\,,\\
\alpha^L_{22} &= \beta^L_{22} = -\alpha^R_{22} = -\beta^R_{22} = 2m+1\,,
\end{aligned} \tag{A.33}$$

where $n,m,k,l\in Z$. Choosing $n=m=k=l=0$, we may get the simplest solution

$$\begin{aligned}
\alpha^L_{11} &= \beta^L_{11} = -\alpha^R_{11} = -\beta^R_{11} = 1\,,\\
\alpha^L_{12} &= \beta^L_{12} = -\alpha^R_{12} = -\beta^R_{12} = 2\,,\\
\alpha^L_{21} &= \beta^L_{21} = -\alpha^R_{21} = -\beta^R_{21} = 0\,,\\
\alpha^L_{22} &= \beta^L_{22} = -\alpha^R_{22} = -\beta^R_{22} = 1\,.
\end{aligned} \tag{A.34}$$

With this solution the boundary condition for $|N,N\rangle$ reads as

$$\psi^i_L|N,N\rangle = i\psi^{i\dagger}_R|N,N\rangle\,,\qquad \psi^{i\dagger}_L|N,N\rangle = i\psi^i_R|N,N\rangle\,,\qquad i=1,2\,. \tag{A.35}$$

Applying $e^{-\pi i J^1_0}$ to $|N,N\rangle$ we obtain the boundary state $|D,N\rangle = e^{-\pi i J^1_0}|N,N\rangle$

$$e^{-\pi i J^1_0}\psi^i_L e^{\pi i J^1_0}|D,N\rangle = i\psi^{i\dagger}_R|D,N\rangle\,. \tag{A.36}$$

Since

$$\begin{aligned}
e^{-\pi i J^1_0}\psi^1_L e^{\pi i J^1_0} &= i\psi^2_L\,,\qquad e^{-\pi i J^1_0}\psi^2_L e^{\pi i J^1_0} = i\psi^1_L\,,\\
e^{-\pi i J^1_0}\psi^{1\dagger}_L e^{\pi i J^1_0} &= -i\psi^{2\dagger}_L\,,q\quad e^{-\pi i J^1_0}\psi^{2\dagger}_L e^{\pi i J^1_0} = -i\psi^{1\dagger}_L\,,
\end{aligned} \tag{A.37}$$

we get the boundary condition for $|D,N\rangle$ in the fermion theory

$$\begin{aligned}
\psi^1_L|D,N\rangle &= \psi^{2\dagger}_R|D,N\rangle\,,\qquad \psi^2_L|D,N\rangle = \psi^{1\dagger}_R|D,N\rangle\\
\psi^{1\dagger}_L|D,N\rangle &= -\psi^2_R|D,N\rangle\,,\qquad \psi^{2\dagger}_L|D,N\rangle = -\psi^1_R|D,N\rangle\,.
\end{aligned} \tag{A.38}$$

The boundary state $|N,D\rangle$ can be obtained by applying $e^{-\pi i J'^1_0}$ to $|N,N\rangle$

$$|N,D\rangle = e^{-\pi i J'^1_0}|N,N\rangle,\quad J'^1_0 = \frac{1}{2}\int\frac{d\sigma}{2\pi}\left(\psi^\dagger_{1L}\psi^\dagger_{2L}+\psi_{2L}\psi_{1L}\right)\,. \tag{A.39}$$

Since

$$e^{-\pi i J_0'^1}\psi_L^1 e^{\pi i J_0'^1} = i\psi_L^{2\dagger}, \quad e^{-\pi i J_0^{1\prime}}\psi_L^2 e^{\pi i J_0'^1} = -i\psi_L^{1\dagger}\,,$$
$$e^{-\pi i J_0^{1\prime}}\psi_L^{1\dagger} e^{\pi i J_0'^1} = -i\psi_L^2, \quad e^{-\pi i J_0^{1\prime}}\psi_L^{2\dagger} e^{\pi i J_0'^1} = i\psi_L^1\,, \tag{A.40}$$

we get the boundary condition for $|N, D\rangle$ in the fermion theory

$$\psi_L^1|N,D\rangle = \psi_R^2|N,D\rangle, \quad \psi_L^2|N,D\rangle = -\psi_R^1|N,D\rangle\,,$$
$$\psi_L^{1\dagger}|N,D\rangle = -\psi_R^{2\dagger}|N,D\rangle, \quad \psi_L^{2\dagger}|N,D\rangle = \psi_R^{1\dagger}|N,D\rangle\,. \tag{A.41}$$

Finally applying $e^{-\pi i(J_0^1+J_0'^1)}$ to $|N, N\rangle$, we get the boundary state $|D, D\rangle$ and its boundary condition. With the help of eq. (A.37) and eq. (A.40)

$$\psi_{1L}|D,D\rangle = i\psi_{1R}|D,D\rangle, \quad \psi_{2L}|D,D\rangle = -i\psi_{2R}|D,D\rangle\,,$$
$$\psi_{1L}^\dagger|D,D\rangle = i\psi_{1R}^\dagger|D,D\rangle, \quad \psi_{2L}^\dagger|D,D\rangle = -i\psi_{2R}^\dagger|D,D\rangle\,. \tag{A.42}$$

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JHEP05(2005)072